

Chapter 8: Ultrasound Obstacle Identification using the Boundary Element and Topological Derivative Methods

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Ultrasound Obstacle Identification using the Boundary Element and Topological Derivative Methods

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Abstract

This chapter gathers together the key concepts of acoustic scattering and the inverse identification problems, the explicit expressions of the topological derivative for sound-hard and penetrable obstacles, and the key features of to solve the inverse identification problem numerically using the Boundary Element Method. Special attention is given to the implementation for the cases in which the search area has pre-existent obstacles, either sound-hard, penetrable or combination of both. Thus, a general BEM implementation framework is provided, which might be used by the reader to develop its own specific application. Several examples for sound-hard obstacles are used to demonstrate the performance of the method.

1 Introduction

Inverse scattering problems find practical applications in the detection and imaging of objects embedded in continuous media, such as the case of ultrasound in medicine, reflection of seismic waves in oil prospecting and crack detection in structural mechanics.

The standard procedure to solve inverse scattering problems consists in emitting waves that interact with the objects and measuring these waves at receptor locations. The total field, consisting of emitted, scattered and transmitted waves, solves a partial differential equation with boundary conditions at the interface between the obstacles and the medium. The inverse problem is stated as follows: knowing the emitted waves and the measured patterns, find the obstacles for which the solution

of the corresponding boundary value problem agrees with the measured at the receptors. Detecting devices like radars, sonars, lidars and scanners follow this general procedure.

The main features of the inverse scattering problem are nonlinearity and ill-posedness. It is nonlinear since the solution to the scattering problem depends nonlinearly on the boundary of the obstacle, and it is ill-posed because given a number of arbitrary measurements at the receptors, a solution for the obstacle boundary may not exist, and if it exists, it may not depend continuously on the parameters that define the obstacle boundary.

Colton and Kress [2013, 2018] provide a review about the historical evolution and the methods to solve inverse scattering problems. The first attempts to solve the inverse obstacle problem dealt with the non-linearity issue by linearizing the problem with the aid of the physical optics approximation. Among others, this approach presents the drawback of being valid only for large wave numbers. Solutions of the full nonlinear inverse obstacle scattering problem can be obtained by means of the so called decomposition methods. These methods break up the problem into two parts: the first part deals with the ill-posedness by constructing the scattered wave from its far field pattern and the second part deals with the nonlinearity by determining the unknown boundary of the obstacle as the location where the boundary condition for the total field is satisfied. However, these methods face the difficulty that in the first step the domain of definition of the scattered wave is not known. Hence, mathematically satisfying formulations of decomposition methods need to combine both parts into an optimization reformulation of the inverse scattering problem.

More modern methods pose the inverse problem as a constrained optimization problem where the 'design variable' is the domain wherein the scattering problem is defined. A cost functional is constructed which quantifies the mismatch between the measured scattering pattern and the scattering pattern corresponding to the current approximation of the shape of the object. The problem is then solved using iterative shape optimization methods, which require, for each step, the computation of the cost function and its derivative with respect to the geometrical parameters that define the boundary of the obstacle, see for example Feijóo [2004] and Feijóo et al. [2004]. This approach is effective, but it does not allow for topological changes in the obstacle, *i.e.*, the number of obstacles has to be known from the beginning. This problem was solved by introducing deformations inspired by level-set methods, which allow creating and destroying boundaries during the iterative process. The paper by Dorn and Lesselier [2006] provides a review on the use of level-set methods for inverse scattering. Nevertheless, level-set based iterative methods may be rather slow unless a good initial guess of the obstacle shape and position is available.

Topological derivative methods have emerged as powerful tools for the numerical solution of inverse problems. The concept firstly appeared in Eschenauer et al. [1994] in connection with topological optimization of mechanical structures, allowing to implement algorithms where 'excess' material is iteratively removed until a satisfactory shape and topology are reached. The basic idea behind the topological derivative is the evaluation of the sensitivity of the cost function towards creating a hole within the problem domain. The topological derivative does not require pre-

existent holes like in level-set methods, so it avoids the aforementioned limitations of shape optimization methods.

The topological derivative to for inverse scattering in acoustics can be traced back to the works by Feijóo [2004], Guzina and Bonnet [2006] and Carpio and Rapún [2008a]. Feijóo [2004] applied the approach introduced by Novotny et al. [2003] to derive topological derivative in a simple and constructive way by using shape sensitivity analysis concepts. On the other hand, Carpio and Rapún [2008a] further developed the topological derivative concept to compute the sensitivity to the insertion of sound-soft obstacles and to deal with non-empty domains.

The boundary element method (BEM) is a very effective numerical technique for solving acoustic problems, especially for simulations that involve infinite domains. One of its advantages is the mesh required. While other techniques like the finite element method (FEM) discretize the entire propagation medium, the discretization of the BEM is limited to the boundary of the objects only. This does not only speed up the model creation and mesh refinement, but it also results in models with less degrees of freedom compared to FEM. The drawback is that unlike FEM, BEM generates fully populated and non-symmetric matrices which limit the benefits of using iterative solvers and memory storage management schemes. Another important aspect of the BEM formulation is that the Sommerfeld's radiation condition (which implies that only outgoing waves are allowed) is implicitly satisfied. Additionally, the unknowns of BEM formulation are the pressure and its derivative, the flux, as such making the method very accurate for the representation of discontinuities Wrobel [2002]. The works by Bonnet [2006], Nemitz and Bonnet [2008], Abe et al. [2010] and Sisamón et al. [2014] are examples of BEM implementations of the topological derivative to inverse acoustic scattering.

This chapter gathers together the key concepts of acoustic scattering and the inverse identification problem (Sections 2, 3 and 4), the explicit expressions of the topological derivative for sound-hard and penetrable obstacles (Section 5), and the key features of to solve the inverse identification problem numerically (Section 6). Special attention is given in Section 6 to the BEM implementation for the cases in which the search area has pre-existent obstacles, either sound-hard, penetrable or combination of both. In this way, a general BEM implementation framework is provided, which might be used by the reader to develop its own specific application. Several examples for sound-hard obstacles are used to demonstrate the performance of the method. Finally, Section 7 is devoted to Comments and Conclusions, with same ideas about further extensions of the methodology to deal with three-dimensional and geometrically complex problems that might require of fast BEM solvers.

2 The Forward Scattering Problem

The general setting of the forward problem is schematized in Figure 1. It consists in an infinite exterior medium Ω_e with a buried obstacle Ω ¹. The illumination of

¹The problem is posed in terms of a single obstacle to preventing the proliferation of subscripts. The extension to multiple obstacles is immediate and will be discussed later in the sections devoted

the medium results in the reflection of the incident radiation, u_{inc} , by the obstacle to produce scattered radiation, u_{sc} . The total radiation field,

$$u = u_{inc} + u_{sc}, \quad (1)$$

is measured along the boundary Γ_{meas} that is placed far enough from the obstacle.

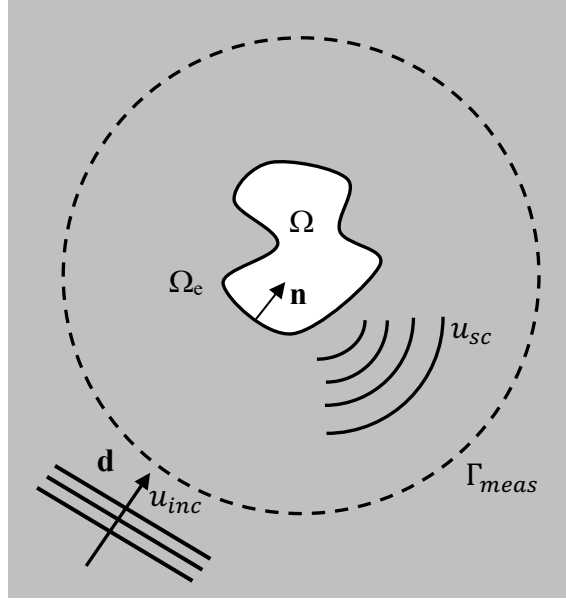


Figure 1: Set-up of the forward scattering problem.

Incident waves can be either planar

$$u_{inc}(\mathbf{x}, \mathbf{d}) = e^{i\lambda_e \mathbf{x} \cdot \mathbf{d}}, \quad (2)$$

where $\mathbf{d}$ is the propagation direction, or due to a point source,

$$u_{inc}(\mathbf{x}, \mathbf{x}_0) = \frac{i}{4} H_0^{(1)}(\lambda_e r), \quad (3)$$

where $\mathbf{x}$ is the evaluation point, $\mathbf{x}_0$ is the location of the point source, i is the imaginary unit, $r = |\mathbf{x} - \mathbf{x}_0|$ is the distance to the source point and $H_0^{(1)}$ is the Hankel function of the first kind. The symbol $\lambda_e = \omega/c$ is the wave number, *i.e.*, the relation of the angular frequency ω to the wave speed c .

The obstacle can be either opaque or penetrable to the incident radiation. An opaque obstacle completely reflects the incident radiation, while for a penetrable obstacle part of the radiation is reflected and the rest is transmitted inside. Thereby, the total wave field in the exterior domain Ω_e and in the interior of the obstacle Ω results after the solution of two coupled Helmholtz problems:

$$\begin{aligned} \nabla^2 u + \lambda_e^2 u &= 0 \text{ in } \Omega_e \text{ and} \\ \alpha \nabla^2 u + \lambda^2 u &= 0 \text{ in } \Omega, \end{aligned} \quad (4)$$

to numerical implementation.

where the coefficient α accounts for the radiation transmitted inside the obstacle –so that $\alpha = 0$ indicates an opaque obstacle– and λ is the wave number for the propagation in the interior of the obstacle. The coupling between the two problems are the transmission conditions on the interface Γ :

$$\begin{aligned} u^- - u^+ &= 0 \text{ and} \\ \alpha \partial_{\mathbf{n}} u^- - \partial_{\mathbf{n}} u^+ &= 0, \end{aligned} \quad (5)$$

where u^+ and u^- are the limits of u from the exterior and interior of Ω , respectively; $\mathbf{n}$ is the unit normal vector pointing outside Ω_e and $\partial_{\mathbf{n}}$ stands for the normal derivative along Γ .

In addition, it is imposed the standard Sommerfeld radiation condition on the propagation of the scattered field u_{sc} at infinity, which implies that only outgoing waves are allowed:

$$\lim_{r \rightarrow \infty} \sqrt{r} (\partial_{\mathbf{r}}(u - u_{inc}) - i\lambda_e(u - u_{inc})) = 0, \quad r = |x|, \quad (6)$$

where $\partial_{\mathbf{r}}$ is the radial derivative and i the imaginary unit.

When the obstacle is opaque there is no transmitted wave inside Ω_e , what allows to reduce the problem to the solution of the external domain Ω_e with a null Neumann boundary condition on Γ . So, the problem in equations (4) to (6) reduces to

$$\begin{aligned} \nabla^2 u + \lambda_e^2 u &= 0 \text{ in } \Omega_e, \\ \partial_{\mathbf{n}} u &= 0 \text{ on } \Gamma \text{ and} \\ \lim_{r \rightarrow \infty} \sqrt{r} (\partial_{\mathbf{r}}(u - u_{inc}) - i\lambda_e(u - u_{inc})) &= 0, \quad r = |\mathbf{x}|. \end{aligned} \quad (7)$$

3 The Inverse Problem

The inverse problem consists in finding the shape and location of the obstacle from the radiation measurements on Γ_{meas} . A variational approach to solve the inverse problem consists in looking for a domain Ω that minimizes the difference between the solution of the forward problem and the measurements on Γ_{meas} . This leads to the following constrained optimization problem: minimize

$$J(\Omega) = \frac{1}{2} \int_{\Gamma_{meas}} |u - u_{meas}|^2 dl. \quad (8)$$

where Ω is the design variable and the forward problem in Section 2 is the constraint on the admissible field u .

When measurements on Γ_{meas} for N different illuminations are available, the optimization problem is: minimize

$$J(\Omega) = \sum_{j=1}^N \frac{1}{2} \int_{\Gamma_{meas}} |u^j - u_{meas}^j|^2 dl. \quad (9)$$

The above problems are strongly ill-posed (Colton [1984]). Given arbitrary data u_{meas} , an associated scatterer Ω may not exist, and if it exists, it may not depend

continuously on u_{meas} . The obstacle is uniquely determined by the far-field scattered wave for all incident directions and one fixed wave number (Kirsch and Kress [1993], Gerlach and Kress [1996]). Therefore, by an analyticity argument (see Colton and Kress [2013]), if Γ_{meas} is a circumference, then the values of the total wave on Γ_{meas} for all incident waves determine uniquely the obstacles. The question of uniqueness without any *a priori* knowledge about the location of the obstacles a finite number of incident plane waves is still an open problem.

4 The Adjoint Problem

As it will be shown in next section, the derivation of the expressions of the topological derivative requires the solution of the so-called adjoint problem. This auxiliary problem consists in propagating back the discrepancy $(u - u_{meas})$ from Γ_{meas} towards the region of analysis.

In the case the region of analysis is empty, the adjoint wave p solves (see Carpio and Rapún [2008a])

$$\begin{aligned} \nabla^2 p + \lambda_e^2 p &= -(u - u_{meas}) \delta_{\Gamma_{meas}} \text{ in } \mathbb{R}^2, \\ \lim_{r \rightarrow \infty} \sqrt{r} (\partial_{\mathbf{r}} p + i \lambda_e p) &= 0, \end{aligned} \quad (10)$$

where $\delta_{\Gamma_{meas}}$ is the Dirac Delta function on Γ_{meas} . For penetrable obstacles, the adjoint problem is

$$\begin{aligned} \nabla p + \lambda_e^2 p &= -(u - u_{meas}) \delta_{\Gamma_{meas}} \text{ in } \Omega_e, \\ \alpha \nabla^2 p + \lambda_e^2 p &= 0 \text{ in } \Omega, \\ p^- + p^+ &= 0 \text{ on } \Gamma, \\ \partial_{\mathbf{n}} p^- + \partial_{\mathbf{n}} p^+ &= 0 \text{ on } \Gamma \text{ and} \\ \lim_{r \rightarrow \infty} \sqrt{r} (\partial_{\mathbf{r}} p + i \lambda_e p) &= 0, r = |\mathbf{x}|. \end{aligned} \quad (11)$$

Finally, for opaque obstacles, the adjoint the problem reduces to the Neumann problem

$$\begin{aligned} \nabla^2 p + \lambda_e^2 p &= -(u - u_{meas}) \delta_{\Gamma_{meas}} \text{ in } \Omega_e, \\ \partial_{\mathbf{n}} p &= 0 \text{ on } \Gamma \text{ and} \\ \lim_{r \rightarrow \infty} \sqrt{r} (\partial_{\mathbf{r}} p + i \lambda_e p) &= 0, r = |\mathbf{x}|. \end{aligned} \quad (12)$$

As for the forward problem, the Sommerfeld radiation condition is imposed to the back-propagated field. In this sense, note the plus sign in the last of equations in (10) to (12) in contrast to the minus sign in the last of equations in (6).

5 The Topological Derivative for Inverse Scattering

The topological derivative measures the sensitivity of the objective function (8) to the inclusion of an infinitesimal obstacle into the problem domain, such that regions where it takes on large negative values are identified as the positions of the obstacles.

The derivation of the topological derivative formulas in this section follows Carpio and Rapún [2008a]. The setting of the problem is in Figure 2. Starting from a domain $\mathcal{R}$ with no obstacles, a small circular obstacle is introduced at a point $\mathbf{x}$, which modifies the domain by creating a 'hole' $B_\varepsilon(\mathbf{x})$ of radius ε . The topological derivative is defined as

$$D_T(\mathbf{x}, \mathcal{R}) = \lim_{\varepsilon \rightarrow 0} \frac{J(\mathcal{R}_\varepsilon) - J(\mathcal{R})}{f(\varepsilon)}, \quad (13)$$

where $J(\mathcal{R})$ and $J(\mathcal{R}_\varepsilon)$ are the cost function evaluated for the reference and perturbed domains, $\mathcal{R}_\varepsilon = \mathcal{R} - B_\varepsilon(\mathbf{x})$, respectively; ε is the radius of the obstacle, and $f(\varepsilon)$ is a monotonically decreasing negative function that is chosen such that $f(\varepsilon) \rightarrow 0$ when $\varepsilon \rightarrow 0$. The function $f(\varepsilon)$ is usually related (although this is not mandatory) to the measure of area of the obstacle, $f(\varepsilon) = -\pi\varepsilon^2$, in this case.

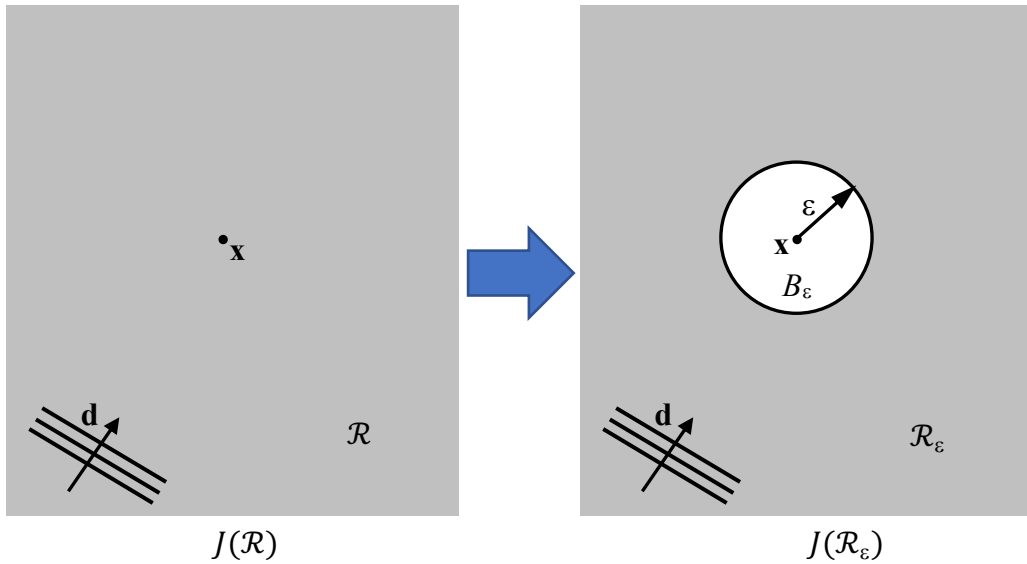


Figure 2: Definition of the topological derivative using the shape sensitivity approach.

Explicit expressions of the topological derivative can be obtained via the shape derivative (Feijóo [2004]). The link between the topological and shape derivatives is established by the relationship

$$J(\mathcal{R}_\varepsilon) = J(\mathcal{R}) + f(\varepsilon)D_T(x) + O(f(\varepsilon)). \quad (14)$$

The shape derivative of the functional (8) along a vector field $\mathbf{V}$ is defined as

$$DJ(\mathcal{R}) \cdot \mathbf{V} = \left. \frac{d}{d\varepsilon} J(\phi(\mathcal{R})) \right|_{\varepsilon=0}, \quad (15)$$

where $\phi(\mathbf{x})$ is the mapping function between the reference and perturbed domains,

$$\phi(\mathbf{x}) = \mathbf{x} + \varepsilon \mathbf{V}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2, \quad (16)$$

in which the vector field points towards the obstacle boundary ∂B , *i.e.*, $\mathbf{V}(\mathbf{x}) = V_n \mathbf{n}(\mathbf{z})$ with $V_n < 0$ vanishes out of a neighborhood of ∂B . Then, the expression for the topological derivative results as

$$D_T(\mathbf{x}, \mathcal{R}) = \lim_{\varepsilon \rightarrow 0} \left(\frac{1}{f'(\varepsilon) |V_n|} \frac{d}{d\varepsilon} J(\phi(\mathcal{R}_\varepsilon)) \Big|_{\varepsilon=0} \right), \quad (17)$$

where $f'(\varepsilon)$ is the derivative of $f(\varepsilon)$.

A family of explicit expressions of the topological derivative results after the solution of (17) for specific problem settings that consider the identification of opaque and penetrable obstacles. For the details about the derivations these solutions – which involve rather technical analyses – the reader is referred to Carpio and Rapún [2008a].

The expressions of the topological derivative for penetrable obstacles is

$$D_T(\mathbf{x}) = Re \left[\frac{2(1 - \alpha)}{1 + \alpha} \nabla u(\mathbf{x}) \nabla \bar{p}(\mathbf{x}) + (\lambda^2 - \lambda_e^2) u(\mathbf{x}) \bar{p}(\mathbf{x}) \right], \quad (18)$$

where $\bar{p}$ stands for the conjugate of the adjoint problem solution. For the limiting case of perfect transmission across the interface, *i.e.* $\alpha = 1$, expression (18) reduces to

$$D_T(\mathbf{x}) = Re \left[(\lambda_i^2 - \lambda_e^2) u(\mathbf{x}) \bar{p}(\mathbf{x}) \right]. \quad (19)$$

Additionally, the topological derivative for opaque obstacles is

$$D_T(\mathbf{x}) = Re \left[2 \nabla u(\mathbf{x}) \nabla \bar{p}(\mathbf{x}) + \lambda_e^2 u(\mathbf{x}) \bar{p}(\mathbf{x}) \right]. \quad (20)$$

It is interesting to point out that the above three expressions for the D_T are independent of the structure of the incident wave, so they hold for plane waves and due to point sources or any other type of source.

Another cool feature of the above expressions is that, although they have been derived starting from an empty domain, they can be used for domains with pre-existent obstacles, say Ω_0 . In such a case, the forward and adjoint problems are solved in $\Omega_e = \mathbb{R}^2 - \Omega_0$ with transmission or Neumann boundary conditions at the interface. The solution of such problems will be discussed thoroughly next in the section devoted to the numerical implementation.

Analogous formulae of the D_T have been deduced in three dimensions, see for example Guzina and Bonnet [2006] and Carpio and Rapún [2008a].

6 Numerical implementation

6.1 Problem setting

For the sake of simplicity, but without loss of generality, the computation of the D_T is presented here in the context of sound scattering and planar incident waves.

The problem setting is depicted in Figure 3. Two cases may arise: either the search area is empty or it contains pre-existent obstacle Ω_0 . In any cases the search

area is illuminated from N different directions and the sound pressure is measured at $\mathbf{x}^k$ with $k = 1 \dots M$ receptor points s that are placed on the circular boundary Γ_{meas} of radius R_{meas} that encloses the search area. As it will be shown later, the use of discrete point receptors to on Γ_{meas} simplifies the computations and allows for efficient BEM implementations.

The D_T is sampled at P regularly spaced points on the search area. These points are referred as the 'domain points' and they are as $\mathbf{x}^l$ with $l = 1 \dots P$.

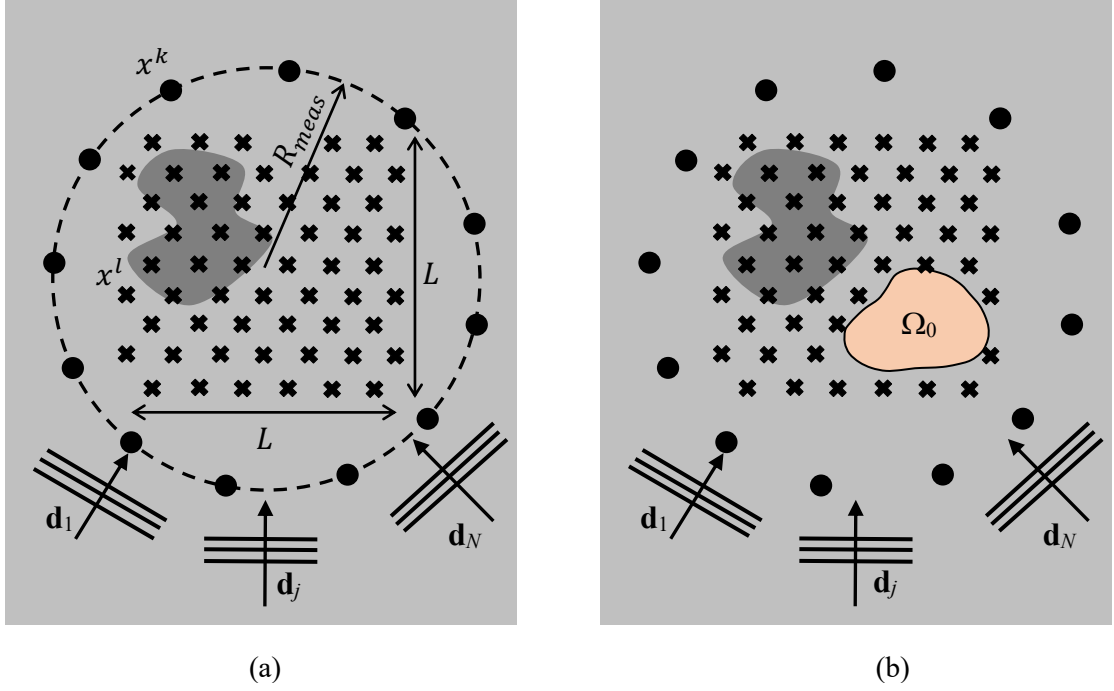


Figure 3: Topological derivative computation: (a) empty domain, (b) domain with a pre-existent obstacle. Symbols $\bullet$ and $\times$ indicate the positions of the receptors and the domain points, respectively.

The evaluation of D_T requires knowing the pressure and its gradient for the direct and adjoint problems (see equations (18), (19) and (20)) at the domains points. There are two implementations: the first for the case of the empty search area, in which all calculations can be performed analytically; and the second for the case of a pre-existing obstacle, which uses numerical solution by means of BEM models.

or it contains pre-existent obstacles.

6.2 Empty domains

The absence of obstacles makes the problem free of dispersion, which allows the pressures and their derivatives of both the direct problem and the adjoint problem to be calculated analytically.

6.2.1 Computation of the direct problem

The incident pressure field is computed using equation (2) for unit incident waves, see Figure 3(a). Due to the absence of scattering, the total pressure is $u = u_{inc}$ (see equation (1)).

On the other hand, the spatial derivatives of the planar wave are

$$u_{,m}(\mathbf{x}, \mathbf{d}) = \frac{\partial u(\mathbf{x}, \mathbf{d})}{\partial x_m} = (i\lambda_e \cdot \mathbf{d}_m) e^{i\lambda_e \mathbf{x} \cdot \mathbf{d}} \text{ with } m = 1, 2. \quad (21)$$

6.2.2 Computation of the adjoint problem

The adjoint problem consist in propagating back the pressure discrepancy a the receptors towards the search area. So, consistent with the point receptors used in this implementation, the radiation from the receptors are computed using equation (3) with the amplitudes given by $(u - u_{meas})$. Thus, the pressure field associated to a point source in the k^{th} receptor is

$$p(\mathbf{x}) = -[u(\mathbf{x}^k) - u_{meas}(\mathbf{x}^k)] \frac{i}{4} H_0^{(1)}(\lambda_e r), \quad (22)$$

and its spatial derivatives are

$$p_{,m}(\mathbf{x}) = -[u(\mathbf{x}^k) - u_{meas}(\mathbf{x}^k)] \frac{i\lambda_e}{4} H_1^{(1)}(\lambda_e r) \frac{\partial r}{\partial x_m} \text{ with } m = 1, 2, \quad (23)$$

where $r = |\mathbf{x} - \mathbf{x}^k|$ is the distance from the receptor.

6.2.3 Algorithm

- i Set-up the problem as it depicted in Figure 3(a), and have available the data of the pressure measurements at the M receptors due to each of the N illumination directions. Store the pressure measurements in $\mathbf{u}_{meas}^j(\mathbf{x}^k)$, which are $j = 1 \dots N$ vectors with $k = 1 \dots M$ elements each.
- ii Use equations (2) and (21) to compute the pressures and pressure gradients due to each of the illumination directions at the domain points. These results are stored in $\mathbf{u}^j(\mathbf{x}^l)$, which are $j = 1 \dots N$ vectors of length $l = 1 \dots P$; and $\nabla \mathbf{u}^j(\mathbf{x}^l)$, which are $j = 1 \dots N$ matrices with $l = 1 \dots P$ rows and two columns, one for each component of the gradient, $m = 1, 2$.
- iii Use equation (2) to compute $\mathbf{u}^j(\mathbf{x}^k)$, the pressures due to each of the $j = 1 \dots N$ illumination directions at the $k = 1 \dots M$ receptors.
- iv Use equations (22) and (23) with the results in $\mathbf{u}_{meas}^j(\mathbf{x}^k)$ and $\mathbf{u}^j(\mathbf{x}^k)$ to compute the adjoint pressure and pressure derivatives at the domain points. Note that since each illumination direction results in discrepancies at all the receptors, the solutions to the adjoint problem at each domain point is the sum of the

contributions of the M receptors. Thus, the adjoint pressure field at the l^{th} domain point do the j^{th} incident wave is

$$\mathbf{p}^j(\mathbf{x}^l) = \sum_{k=1}^M -[\mathbf{u}^j(\mathbf{x}^k) - \mathbf{u}_{\text{meas}}^j(\mathbf{x}^k)] \frac{i}{4} H_0^{(1)}(\lambda_e r^{(k)}), \quad (24)$$

and its spatial derivatives are

$$\mathbf{p}_{,m}^j(\mathbf{x}^l) = \sum_{k=1}^M -[\mathbf{u}^j(\mathbf{x}^k) - \mathbf{u}_{\text{meas}}^j(\mathbf{x}^k)] \frac{i\lambda_e}{4} H_1^{(1)}(\lambda_e r^{(k)}) \frac{\partial r^{(k)}}{\partial x_m} \text{ with } m = 1, 2. \quad (25)$$

Following the same scheme in step (ii), the results of the above computations are stored in the arrays $\mathbf{p}^j(\mathbf{x}^l)$ and $\nabla \mathbf{p}^j(\mathbf{x}^l)$, respectively.

- v Select the adequate expression of D_T in Section 5 and use the results in $\mathbf{u}^j(\mathbf{x}^l)$, $\nabla \mathbf{u}^j(\mathbf{x}^l)$, $\mathbf{p}^j(\mathbf{x}^l)$ and $\nabla \mathbf{p}^j(\mathbf{x}^l)$, to compute the topological derivative at the domain points for each of the N illumination directions: $\mathbf{D}_{\mathbf{T}}^j(\mathbf{x}^l)$.
- vi Compute the overall topological derivative at the domain points as the sum of the contributions of all the illumination directions:

$$D_T(\mathbf{x}^l) = \sum_{j=1}^N \mathbf{D}_{\mathbf{T}}^j(\mathbf{x}^l) \quad (26)$$

- vii Draw a contour plot with the $D_T(\mathbf{x}^l)$ results over the search domain. The geometry of the hidden obstacle is that delimited by the largest negative values of the topological derivative.

6.2.4 Examples

Identification of a single obstacle. The hidden obstacle is that depicted in Figure 4a, which has a smooth boundary with concave and convex portions.

The problem set-up is similar to that depicted in Figure 3a. The search area is of dimensions $L \times L = 10 \text{ m} \times 10 \text{ m}$. The hidden obstacle is illuminated by N plane waves with their angles of incidence equally distributed over 2π . The number of receptors M , the distance R_{meas} , as well as the number of illumination waves N and their wavelengths λ are subjects of the analysis. The reference sound-pressure values at the receptors due to the dispersion of the N incident waves by the obstacle, $\mathbf{u}_{\text{meas}}^j(\mathbf{x}^k)$, were produce synthetically using high-resolution BEM models with element sizes equal to one-tenth the incident wavelength (the details of the BEM models will be introduced later in Section 6.3).

The first set of results assess the effect of the wavelength on the quality of the identification. The analysis is performed for wavelengths $\lambda = c/f_i = 3.4, 1.7, 0.85, 0.567$ and 0.425 m , corresponding to frequencies $f_i = 100, 200, 400, 600$ and 800 Hz for a speed of sound $c = 340 \text{ m/s}$. The number of incident waves, $N = 100$, the number of receptors, $M = 100$, and their distances, $R_{\text{meas}} = 10 \text{ m}$, are kept constant.

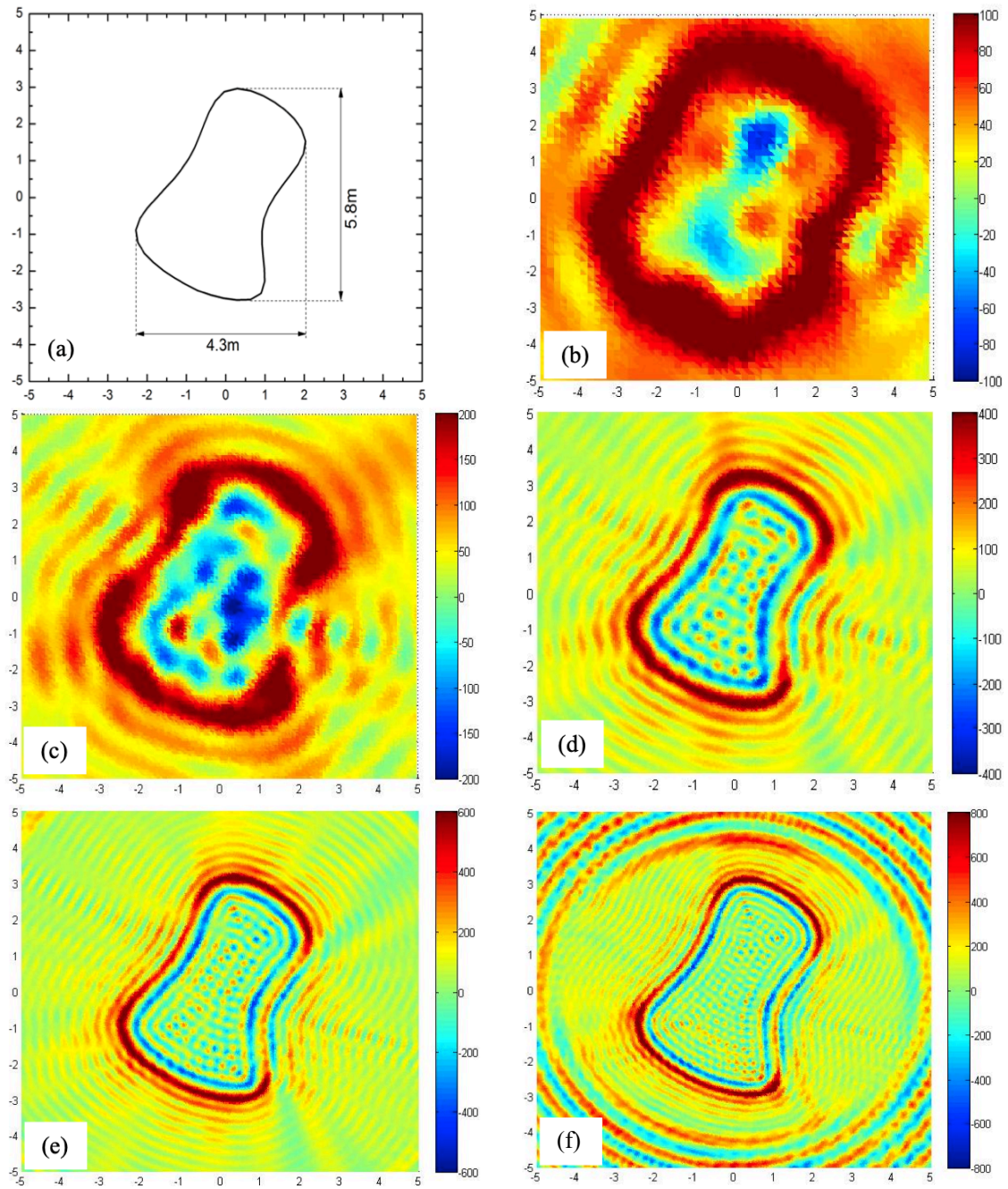


Figure 4: Identification of a single obstacle in an empty domain. Contour plots of D_T for different illumination wavelengths: (a) Geometry of the hidden obstacle and results for the obstacle shape for (b) $\lambda/D = 0.85$, (c) $\lambda/D = 0.425$, (d) $\lambda/D = 0.213$, (e) $\lambda/D = 0.142$ and (f) $\lambda/D = 0.106$.

Grid size of the domain point arrangement are one-tenth of the wavelength of the incident wave. Depending on the wavelength, the number of domain points ranged from 870 to 55360.

The results for the topological derivative D_T in terms of λ/D , the ratio between the wavelength and the characteristic length of the obstacle, are presented in Figures 4b to f. The shape of the scatterer is given by the locus of the most negative lowest D_T values. As it was expected, the quality of the identification improves as the wavelength decreases. The reconstruction provides a reasonable result of the obstacle shape for wavelengths $\lambda/D \leq 0.20$.

The second analysis is about the influence of the number of illumination directions. The problem is solved for $N_i = 25, 50, 75$ and 100. Based on the previous analysis, the source wavelength is set to $\lambda/D = 0.142$ ($f = 600Hz$). The number and distance of the receptors, $M = 100$ and $R_{meas} = 10$ m, are the same as in the previous analysis. The results are shown in Figure 5. It can be observed that a reduced number of sources as low as $N = 25$ provides a reasonable approximation of the obstacle shape.

The effect of the distance of the receptors is assessed in the third analysis. To this end, the problem is solved for the normalized distances $R_{meas}/D = 2.5, 5, 10$ and 20. The number of illumination directions, their wavelengths and the number of receptors are set to $N = 100$, $\lambda/D = 0.142$ and $M = 100$, respectively. The results are shown in Figure 6. As it was expected, the quality of the identification deteriorates with the distance increases. However, this lost of performance is not significant for the range of distances considered.

Finally, the effect of the measurement error is investigated by adding noise to the pressure values at the receptors. Random noise is added to the amplitude and phase of $u_{meas_k}^j$. The effects of the noise amplitude are examined in the range from 5% to 50% the amplitude of the incident waves. Figure 7 illustrates some results for the example with $N = 100$, $\lambda/D = 0.142$, $M = 100$ and $R_{meas}/D = 10$. The effect of noise starts to be noticeable for noise amplitudes around 20% (compare Figures 6c and 7a). Noise amplitudes of 50% clearly deteriorates the performance of the method, see Figure 7b; however, it is still possible to distinguish the silhouette of the hidden obstacle.

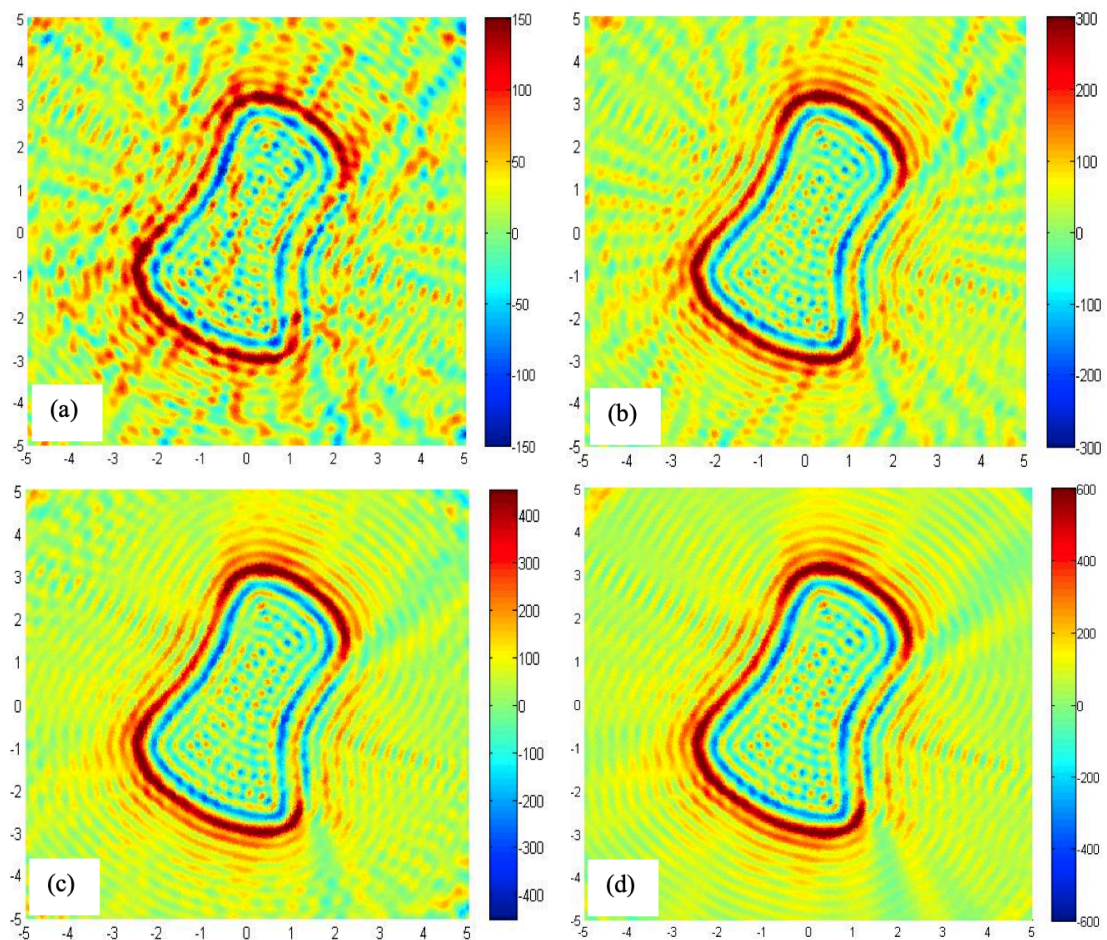


Figure 5: Identification of a single obstacle in an empty domain. Contour plots of D_T in terms of the number of sources: (a) $N = 25$, (b) $N = 50$, (c) $N = 75$ and (d) $N = 100$.

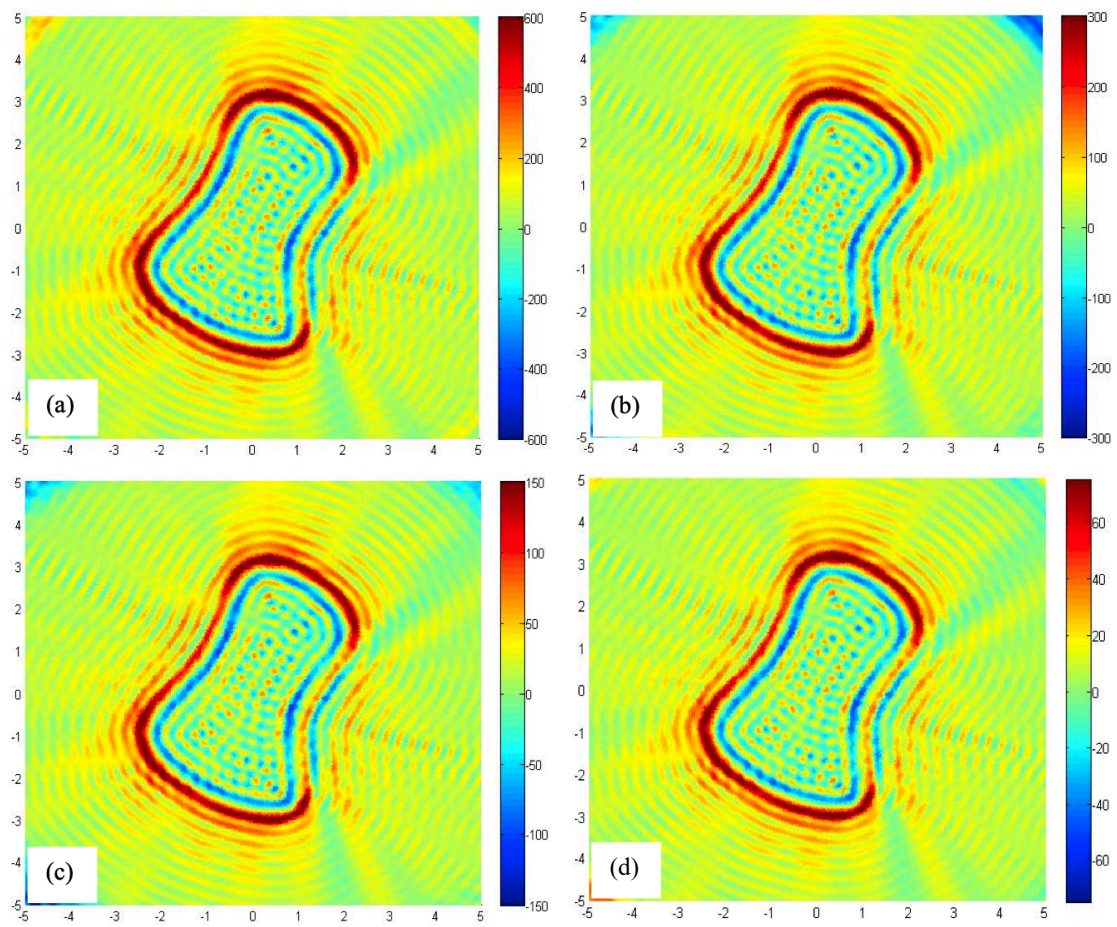


Figure 6: Identification of a single obstacle in an empty domain. Contour plots of D_T in terms of the distance to the receptors: (a) $R_{meas}/D = 2.5$, (b) $R_{meas}/D = 5$, (c) $R_{meas}/D = 10$ and (d) $R_{meas}/D = 20$.

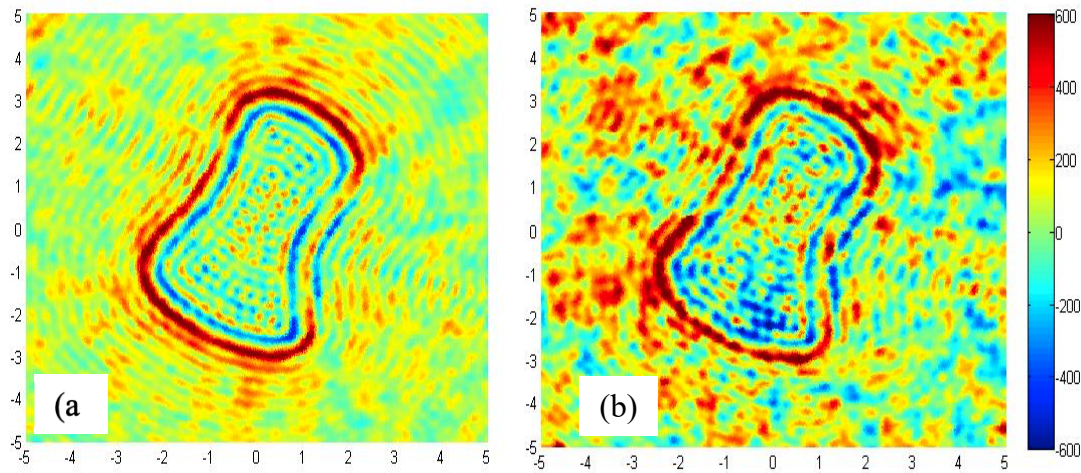


Figure 7: Identification of a single obstacle in an empty domain. Contour plots of D_T in terms of the number of sources: (a) $N = 25$, (b) $N = 50$, (c) $N = 75$ and (d) $N = 100$.

Identification of multiple obstacles The problem geometry is illustrated in 8a. It consists in the identification of two hidden obstacles: a circle of radius $r = 2 \text{ m}$ and a square with side length $s = 2r = 4 \text{ m}$.

The performance of the method is demonstrated with regard to the distance between the objects, which is examined in the range $2\sqrt{2} \leq d/r \leq 6\sqrt{2}$, where d is the distance from the center of the circle to the center of the square. Based on the results of the previous example, $M = 100$ receptors are placed on the circular boundary Γ_{meas} of radius $R_{meas} = 10r = 20 \text{ m}$. The search area is illuminated by $N = 100$ planar waves with their angles of incidence equally distributed over 2π . The wavelength is set to $\lambda = 0.567 \text{ m}$, so that $\lambda/s = 0.142$. The size of the search area is set in accordance with the distance of the scatterers, from $12 \text{ m} \times 12 \text{ m}$ for the closest positions to $20 \text{ m} \times 20 \text{ m}$ for the furthest one. Accordingly, the number of domain points ranges from 58081 to 161336. Like in the previous example, reference sound-pressure values at the receptors due to the hidden scatterers, $u_{meas_k}^j$, were computed using high-resolution BEM models.

The D_T results in terms of distance between the obstacles s are plotted in Figure 8b to f. It can be observed that obstacles shapes are resolved with high precision. It is interesting to note that the D_T intensities along the portions of the obstacles boundaries facing each other are larger than along the portions of the boundaries facing the open space. This is because the obstacles shield each other, what hinders the illumination of the closest portions of the boundaries. Moreover, the highest values of D_T occur in the zone between the obstacles, what indicates that this is the most sensitive zone towards the minimization of the objective function. This is coherent with the behavior expected for the limiting case, in which the objects touch or overlap, and thus behave as a single obstacle.

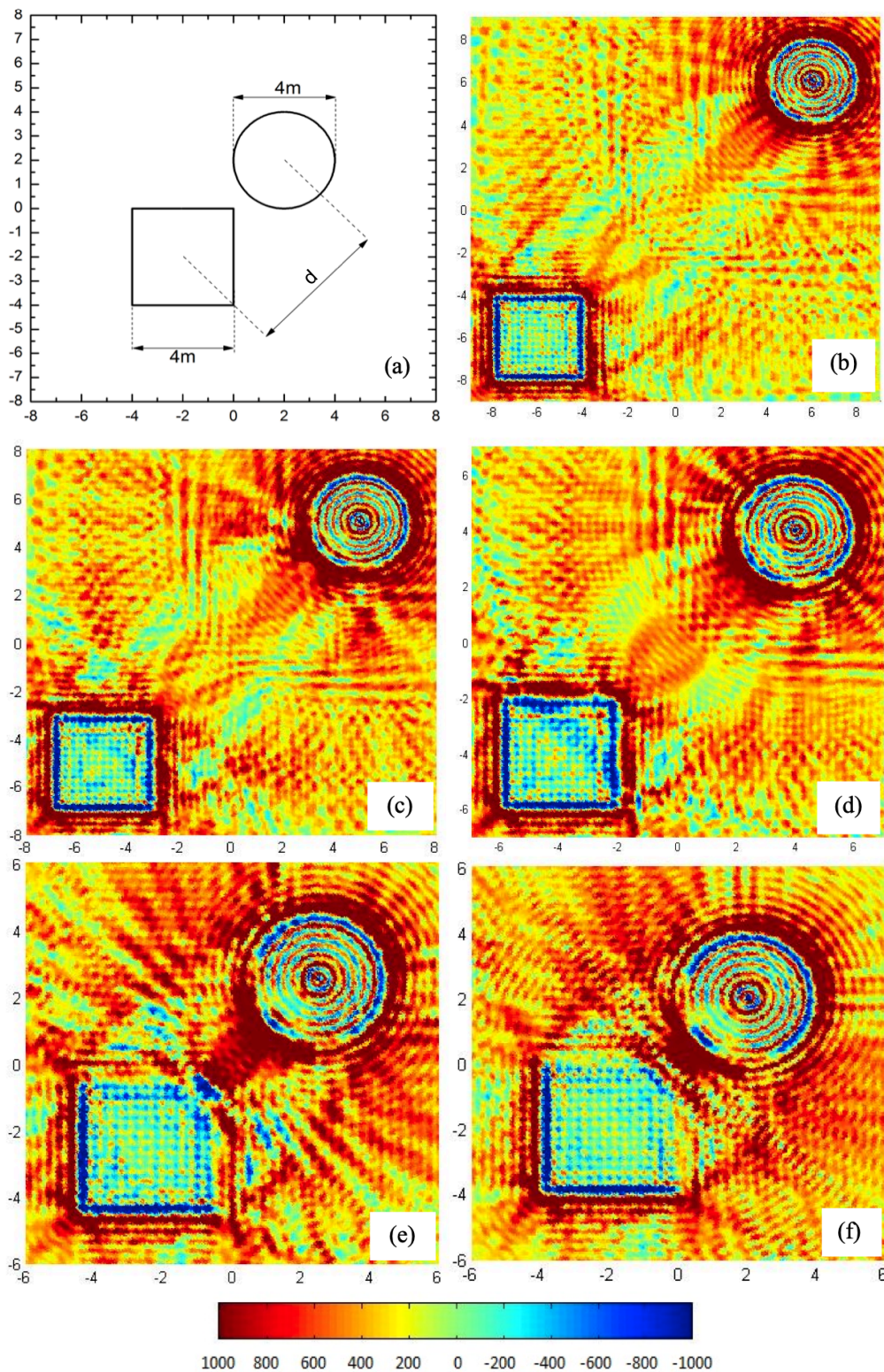


Figure 8: Identification of two obstacles in an empty domain. Contour plots of D_T in terms of the separation between the obstacles: (a) hidden obstacles, (b) $d/r = 6\sqrt{2}$, (c) $d/r = 5\sqrt{2}$, (d) $d/r = 4\sqrt{2}$, (e) $d/r = 2.5\sqrt{2}$ and (f) $d/r = 2\sqrt{2}$. The same color scale is used in all the figures.

6.3 Domains with pre-existent obstacles

In contrast to the empty-domain analyses, the evaluation of the D_T for domains with pre-existent obstacles requires of numerical models to solve the scattered fields of the forward and adjoint problems. Boundary element models are used for this purpose.

6.3.1 The BEM for the Helmholtz's equation

Main aspects of the Boundary Element Method (BEM) are described next, aiming to the specific task of computing the D_T . The BEM formulation is not introduced thoroughly; the reader is referred to classical books on BEM such as Wrobel [2002] or Brebbia et al. [1984] for full details.

Consider the problem in Figure 3(b) and assimilate Ω_0 to a hole, such that $\Omega_e = \mathcal{R} - \Omega_0$. The starting point to formulate the BEM for solving the scatter problem over Ω_e is the integral formulation of the Helmholtz equation:

$$c(\mathbf{x}')u_{sc}(\mathbf{x}') + \int_{\Gamma} q^*(\mathbf{x}, \mathbf{x}')u_{sc}(\mathbf{x})d\Gamma(\mathbf{x}) = \int_{\Gamma} u^*(\mathbf{x}, \mathbf{x}')q_{sc}(\mathbf{x})d\Gamma(\mathbf{x}) + u_{inc}(\mathbf{x}', \mathbf{d}), \quad (27)$$

which relates $u_{sc}(\mathbf{x}')$, the pressure at a position $\mathbf{x}'$ (the so-called collocation point), with the pressure and the pressure flux, $u_{sc}(\mathbf{x}')$ and $q_{sc}(\mathbf{x}) = \frac{\partial u_{sc}}{\partial \mathbf{n}}$, respectively, on the boundary $\Gamma(\mathbf{x})$. The term $c(\mathbf{x}')$ in (27) is a coefficient that depends on the boundary geometry at $\mathbf{x}'$; and $u^*(\mathbf{x}, \mathbf{x}')$ and $q^*(\mathbf{x}, \mathbf{x}')$ are the fundamental solutions for the pressure and flux fields, respectively.

The pressure fundamental solution u^* is the solution to the Helmholtz's equation for the pressure generated at a field point $\mathbf{x}$ by a concentrated unit source at $\mathbf{x}'$ in an unbounded space. Note that this solution is the same of the point source in equation (3). So,

$$u^*(\mathbf{x}, \mathbf{x}') = \frac{i}{4}H_0^{(1)}(\lambda_e r), \quad (28)$$

with the distance $r = |\mathbf{x} - \mathbf{x}'|$ defined from the collocation to the field point. Similarly, the fundamental solution q^* is the flux in the normal direction at $\Gamma(\mathbf{x})$ generated by the unit source at $\mathbf{x}'$, this is

$$q^*(\mathbf{x}, \mathbf{x}') = \frac{\partial u^*}{\partial \mathbf{n}} = -\frac{i\lambda_e}{4}H_0^{(1)}(\lambda_e r). \quad (29)$$

It is worth to note that the above formulation naturally satisfies the standard Sommerfeld radiation condition on the propagation of the scattered field at infinity (see Section 2).

The BEM consists in discretizing the boundary $\Gamma(\mathbf{x})$ into nel elements. Depending on the discretization strategy, $u(\mathbf{x})$ and $q(\mathbf{x})$ can be interpolated using constant, linear or quadratic elements. For the sake of simplicity but without the loss of generality, constant elements² are used in this work, such that $\Gamma = \sum_{j=1}^{nel} \Gamma_j$. The

²These elements approximate both the u and q fields as constants and have a single node at their midpoints.

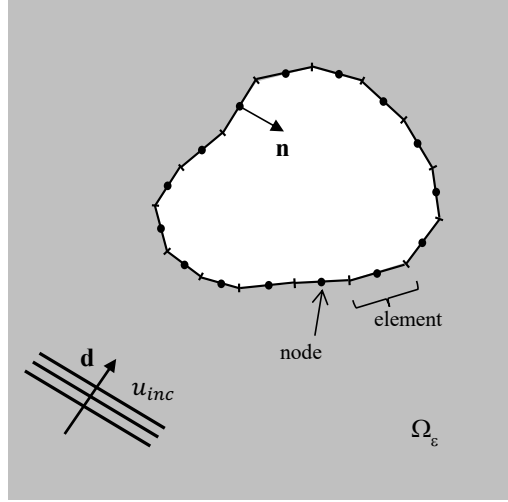


Figure 9: Boundary element discretization of the problem consisting in an open domain with an embedded hole.

discretized model is depicted in Figure 9. Notice the normal vector pointing out of the domain. The discretized version of equation (27) for the collocation point located at the i^{th} node is

$$c^i(\mathbf{x}^i)u_{sc}^i(\mathbf{x}^i) + \sum_{j=1}^{nel} \int_{\Gamma_j} u_{sc}^j(\mathbf{x})q^*(\mathbf{x}, \mathbf{x}^i)d\Gamma(\mathbf{x}) = \sum_{j=1}^{nel} \int_{\Gamma_j} q_{sc}^j(\mathbf{x})u^*(\mathbf{x}, \mathbf{x}^i)d\Gamma(\mathbf{x}) + u_{inc}(\mathbf{x}^i, \mathbf{d}). \quad (30)$$

Upon the evaluation the of the integrals, equation (30) can be written in matrix form as follows:

$$c^i u_{sc}^i + \sum_{j=1}^{nel} \hat{H}^{ij} u_{sc}^j = \sum_{j=1}^{nel} \int_{\Gamma_j} G^{ij} q_{sc}^j + u_{inc}^i, \quad (31)$$

where

$$\begin{aligned} \hat{H}^{ij} &= \int_{\Gamma_j} q^*(\mathbf{x}, \mathbf{x}^i)d\Gamma(\mathbf{x}) \text{ and} \\ G^{ij} &= \int_{\Gamma_j} u^*(\mathbf{x}, \mathbf{x}^i)d\Gamma(\mathbf{x}). \end{aligned} \quad (32)$$

If the position of the collocation point i is made to vary from 1 to nel , that is, the fundamental solution is applied in each of the nodes successively, the evaluation of (31) to each node results in the system of equations

$$\mathbf{H}\mathbf{u}_{sc} = \mathbf{G}\mathbf{q}_{sc} + \mathbf{u}_{inc}, \quad (33)$$

where $\mathbf{H}$ collects the elements $\hat{H}^{ij}$ plus the terms $c^i(\mathbf{x}^i)$.

Matrices $\mathbf{H}$ and $\mathbf{G}$ in equation (33) have dimensions $nel \times nel$, while $\mathbf{u}_{sc}$, $\mathbf{q}_{sc}$ and $\mathbf{u}_{inc}$ are vectors of length nel . This makes a system of nel equations with

$2 \cdot nel$ unknowns (u_{sc}^i and q_{sc}^i), which will not be solvable until boundary conditions are specified on Γ . In the next section, the formulation will be specialized for the cases of pre-existing sound-hard (opaque) and penetrable obstacles by specifying the appropriate boundary conditions on Γ .

6.3.2 Computation of the forward problem

Sound hard obstacles. When dealing with sound hard obstacles, the sound flux $q = \frac{\partial u_{sc}}{\partial \mathbf{n}}$ is zero along $\Gamma(\mathbf{x})$. So, the integrals in the right-hand side of (30) vanish to yield

$$c^i u_{sc}^i + \sum_{j=1}^{nel} \hat{H}^{ij} u_{sc}^j = u_{inc}^i, \quad (34)$$

and the system of equation (33) reduces to

$$\mathbf{H} \mathbf{u}_{sc} = \mathbf{u}_{inc}. \quad (35)$$

The system in (35) is solved to obtain the nodal pressures on the obstacle boundary, $\mathbf{u}_{sc}$. Notice that matrix $\mathbf{H}$ depends on the geometry of the obstacle only, so, it is computed once and then used repeatedly to solve the boundary pressures for each of the N incident waves. Conversely, vector $\mathbf{u}_{inc}$ depends on the direction of the incident wave, so it has to be computed repeatedly for each incident wave.

Once the boundary pressure is known, the total pressures and pressure gradients at the domain points are computed using equation (1) in combination with the pressure boundary integral equation. Thus, the discretized expressions for the total pressure and its gradient components are

$$u_{sc}(\mathbf{x}^l) = \sum_{j=1}^{nel} \int_{\Gamma_j} u_{sc}^j(\mathbf{x}) q^*(\mathbf{x}, \mathbf{x}^l) d\Gamma(\mathbf{x}) + u_{inc}(\mathbf{x}^l, \mathbf{d}) \quad (36)$$

and

$$u_{sc,m}(\mathbf{x}^l) = \frac{\partial u_{sc}(\mathbf{x}^l)}{\partial \mathbf{x}_m^l} = \sum_{j=1}^{nel} \int_{\Gamma_j} u_{sc}^j(\mathbf{x}) \frac{\partial q^*(\mathbf{x}, \mathbf{x}^l)}{\partial \mathbf{x}_m} d\Gamma(\mathbf{x}) + \frac{\partial u_{inc}(\mathbf{x}^l, \mathbf{d})}{\partial \mathbf{x}_m} \quad \text{with } m = 1, 2, \quad (37)$$

respectively. The discretized matrix forms of equations (36) and (37) are

$$\mathbf{u}_{sc}(\mathbf{x}^l) = \mathbf{A} \mathbf{u}_{sc} + \mathbf{u}_{inc} \quad (38)$$

and

$$\mathbf{u}_{sc,m}(\mathbf{x}^l) = \mathbf{A}' \mathbf{u}_{sc} + \mathbf{u}_{inc,m}. \quad (39)$$

Notice that matrices $\mathbf{A}$ and $\mathbf{A}'$ have to be computed repeatedly for each domain node, since they depend on the position $\mathbf{x}^l$. The computation of $\mathbf{A}$ and $\mathbf{A}'$ are the most expensive tasks of the algorithm, as they are typically computed hundreds or thousands of times.

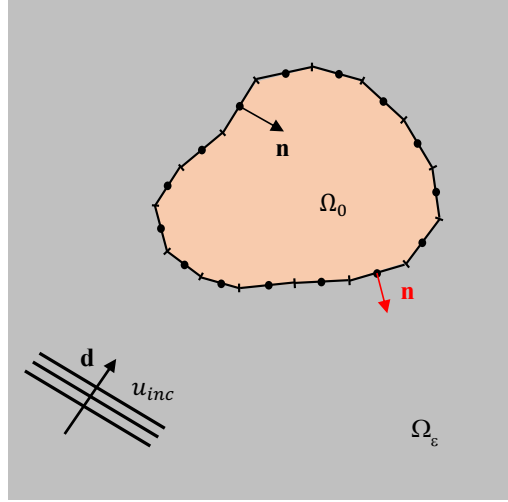


Figure 10: Boundary element discretization of the problem with a pre-existent obstacle.

It is also worth to notice that although the above formulation has been introduced for a single pre-existent obstacle, its extension to multiple obstacles is immediate. In such a case, it is only necessary to assimilate Γ to the union of boundaries of all obstacles.

In turn, expression (36) is also used to compute the sound pressures at the receptors, $\mathbf{u}(\mathbf{x}^k)$, which will later used to pose the adjoint problem.

Penetrable obstacles. The BEM setup for penetrable obstacles needs of multiple domains in order to account for the propagation within the obstacles. For the sake of simplicity, a formulation for a single pre-existent obstacle will be introduced next; its extension to multiple pre-existing obstacles is straightforward.

Two domains are considered: the external infinite domain, Ω_e , and that of an embedded pre-existent obstacle, Ω_0 , as in Figure 3(b). A simple strategy for developing the general BEM model is to apply the boundary element procedure to each domain independently, and then combine their systems of equations using the compatibility and continuity conditions at the interface.

As shown in Figure 10, the two domains use the same boundary element mesh, but they have opposite normal vectors (this is because both are external normals of their corresponding domains). The integral formulation for the exterior problem is that in equation (27). Analogously, for the embedded domain is

$$c^0(\mathbf{x}')u_{sc}^0(\mathbf{x}') + \int_{\Gamma_0} q_0^*(\mathbf{x}, \mathbf{x}')u_{sc}^0(\mathbf{x})d\Gamma(\mathbf{x}) = \int_{\Gamma_0} u_0^*(\mathbf{x}, \mathbf{x}')q_{sc}^0(\mathbf{x})d\Gamma(\mathbf{x}), \quad (40)$$

where fundamental solutions u_0^* and q_0^* are the same in (28) and (29), respectively, but with λ_e replaced by λ_0 .

The discretized matrix form of equation (40) is

$$\mathbf{H}_0 \mathbf{u}_{sc}^0 = \mathbf{G}_0 \mathbf{q}_{sc}^0. \quad (41)$$

If one adopts the pressure and the flux of Ω_e as reference values (which is equivalent to say that the normal on the interface is the normal to Ω_e), the compatibility and equilibrium conditions in equations (5) results in

$$\mathbf{u}_{sc} = \mathbf{u}_{sc}^0 \quad (42)$$

and

$$\mathbf{q}_{sc} = -\alpha \mathbf{q}_{sc}^0. \quad (43)$$

These conditions can be introduced in (33) and (41), which can now be written together as follows:

$$\begin{bmatrix} \mathbf{H} \\ \mathbf{H}_0 \end{bmatrix} \mathbf{u}_{sc} = \begin{bmatrix} \mathbf{G} \\ -\alpha \mathbf{G}_0 \end{bmatrix} \mathbf{q}_{sc} + \begin{bmatrix} \mathbf{u}_{inc} \\ \mathbf{0} \end{bmatrix} \quad (44)$$

The system above can be rewritten as

$$\begin{bmatrix} \mathbf{H} & -\mathbf{G} \\ \mathbf{H}_0 & \alpha \mathbf{G}_0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_{sc} \\ \mathbf{q}_{sc} \end{bmatrix} = \begin{bmatrix} \mathbf{u}_{inc} \\ \mathbf{0} \end{bmatrix}, \quad (45)$$

where the matrix on the left-hand side has dimensions $2 \cdot nel \times 2 \cdot nel$ and the vector containing $\mathbf{u}_{sc}$ and $\mathbf{q}_{sc}$ is of length $2 \cdot nel$. The system is then solved to obtain the pressures and fluxes on the obstacle boundary.

The above formulation can be easily extended to handle multiple obstacles, which could have different penetration coefficients α . Moreover, sound-hard and penetrable obstacles can be combined into a single BEM scheme. It is also interesting to point out that when more obstacles are included, the system of equations (45) tend to have large number of zero submatrices, which improves computational efficiency.

Once pressures and fluxes on the obstacle boundary are known, total pressures and pressure gradients at the domain points in Ω_e are computed using their boundary integral representations. Their discretized expressions are

$$u_{sc}(\mathbf{x}^l) = \sum_{j=1}^{nel} \int_{\Gamma_j} u_{sc}^j(\mathbf{x}) q^*(\mathbf{x}, \mathbf{x}^l) d\Gamma(\mathbf{x}) + \sum_{j=1}^{nel} \int_{\Gamma_j} q_{sc}^j(\mathbf{x}) u^*(\mathbf{x}, \mathbf{x}^l) d\Gamma(\mathbf{x}) + u_{inc}(\mathbf{x}^l, \mathbf{d}) \quad (46)$$

and

$$\begin{aligned} u_{sc,m}(\mathbf{x}^l) &= \frac{\partial u_{sc}(\mathbf{x}^l)}{\partial \mathbf{x}_m^l} = \sum_{j=1}^{nel} \int_{\Gamma_j} u_{sc}(\mathbf{x}) \frac{\partial q^*(\mathbf{x}, \mathbf{x}^l)}{\partial \mathbf{x}_m} d\Gamma(\mathbf{x}) + \sum_{j=1}^{nel} \int_{\Gamma_j} q_{sc}(\mathbf{x}) \frac{\partial u^*(\mathbf{x}, \mathbf{x}^l)}{\partial \mathbf{x}_m} d\Gamma(\mathbf{x}) + \\ &+ \frac{\partial u_{inc}(\mathbf{x}^l, \mathbf{d})}{\partial \mathbf{x}_m} \text{ with } m = 1, 2. \end{aligned} \quad (47)$$

Notice that in comparison to the sound-hard case, the above equations incorporate an extra term that accounts for the boundary flux. The discretized matrix forms of equations (46) and (47) are

$$\mathbf{u}_{sc}(\mathbf{x}^l) = \mathbf{A} \mathbf{u}_{sc} + \mathbf{B} \mathbf{q}_{sc} + \mathbf{u}_{inc} \quad (48)$$

and

$$\mathbf{u}_{sc,m}(\mathbf{x}^l) = \mathbf{A}'\mathbf{u}_{sc} + \mathbf{B}'\mathbf{q}_{sc}\mathbf{u}_{inc,m}. \quad (49)$$

The comments for the reusability of $\mathbf{A}$, $\mathbf{A}'$, $\mathbf{B}$ and $\mathbf{B}'$ and the most efficient implementation strategy are the same as for the sound-hard case.

6.3.3 Computation of the adjoint problem

The solution of the adjoint problem in Section 4 is decomposed –as it is done for the forward problem– into the incident and scattered parts, *i.e.*, $p = p_{inc} + p_{sc}$.

- The solutions for the incident pressure and the pressure gradient are the same of forward problem in Section 6.2.2. So, p_{inc} and $p_{inc,m}$ fields due to the point source in the k_{th} receptor are computed using equations (22) and (23), respectively.
- The solution of the scattered field is analogous to the forward problem in Section 6.3.2, so p_{sc} and $p_{sc,m}$ can be computed using the same BEM schemes. For this, it is only necessary to specify the incident waves as point sources of amplitude $-[u(\mathbf{x}^k) - u_{meas}(\mathbf{x}^k)]$ at the receptors. Note that all BEM matrices (those to solve the boundary problem and to compute the solutions at the internal points) are the same of the forward problem, so they do not need to be recomputed.

6.3.4 Algorithm

The algorithm for the evaluation of the D_T is basically the same introduced in Section 6.2.3 for the empty space. However, it needs to be adapted to replace the analytical solutions by BEM ones.

- Set-up the problem as it depicted in Figure 3(b), and have available the data of the pressure measurements at the M receptors due to each of the N illumination directions. Store the pressure measurements in $\mathbf{u}_{meas}^j(\mathbf{x}^k)$.
- Use the BEM to solve the forward problem for each of the illumination directions and compute the corresponding pressures, $\mathbf{u}^j(\mathbf{x}^l)$, and pressure gradients, $\nabla\mathbf{u}^j(\mathbf{x}^l)$, at the P domain points. For this, choose the appropriate BEM formulation from Section 6.3.2, depending on whether the pre-existent obstacles are sound-hard or penetrable.
- Use the BEM model to compute the pressures $\mathbf{u}^j(\mathbf{x}^k)$ at the M receptors for each of the illumination directions.
- Follow the strategy in Section 6.3.3 to use the BEM model and the discrepancies at the receptors, $[u(\mathbf{x}^k) - u_{meas}(\mathbf{x}^k)]$, to calculate the adjoint pressure and pressure derivatives at the domain points for each illumination direction. Then, add up the contributions of the M receptors to compute $\mathbf{p}^j(\mathbf{x}^l)$ and $\nabla\mathbf{p}^j(\mathbf{x}^l)$, the adjoint fields at the domain points for each illumination direction.

- v Select the adequate expression of D_T in Section 5 and use the results in $\mathbf{u}^j(\mathbf{x}^l)$, $\nabla \mathbf{u}^j(\mathbf{x}^l)$, $\mathbf{p}^j(\mathbf{x}^l)$ and $\nabla \mathbf{p}^j(\mathbf{x}^l)$, to compute the topological derivative at the domain points for each illumination directions: $\mathbf{D}_T^j(\mathbf{x}^l)$.
- vi Compute $D_T(\mathbf{x}^l)$, the overall topological derivative at the domain points as the sum of the contributions of all the illumination directions.
- vii Draw a contour plot with the $D_T(\mathbf{x}^l)$ results over the search domain. The geometry of the hidden obstacle is that delimited by the largest negative values of the topological derivative.

6.3.5 Example

This example revisits the problem of the multiple obstacle identification in Section 6.2.4, but with the variation that square obstacle is visible while the circular one is hidden.

The problem is solved using the same wavelength, number and position of the receptors and number of incident waves as in the previous analysis. The BEM models for the computation of the forward and adjoint problems are discretized with elements of size equal to one-tenth of the wavelength.

The D_T results in terms of distance between the obstacles are plotted in Figure 11. Plots show the effectiveness of the procedure to identify the circular obstacle. It is interesting to see that the quality of the identification of the circle, especially the portion of the boundary that faces the square, is better than in the previous analysis. This result because, being the presence of the square known *a priori*, the identification does not suffer the shielding effects when the obstacles are close to each other.

The robustness of the method to measurement error is investigated using the same strategy as for the previous example. The case $d/r = 5\sqrt{2}$ in Figure 11b, is selected for the analysis. Figure 12 illustrates some results. It is found that the degradation of the results starts to be noticeable for noise amplitudes around 10 %, see Figure 10a. However, and like in the previous example, it is still possible to identify the hidden object with measurement errors of up to around 50%, , see Figure 12b.

7 Comments and Conclusions

A framework for the inverse analysis of acoustic scattering problems in open domains using the Boundary Element and Topological Derivative Methods has been presented in the previous sections. The modeling strategy takes advantage of the inherent characteristics of the BEM to effectively deal with problems with infinite domains. The tool has the capability to identify obstacles in initially empty domains and as well as in domains with pre-existent obstacles.

The performance of the tool is demonstrated for sound-hard obstacles. Such problems are solved using single-domain BEM models. An efficient BEM imple-

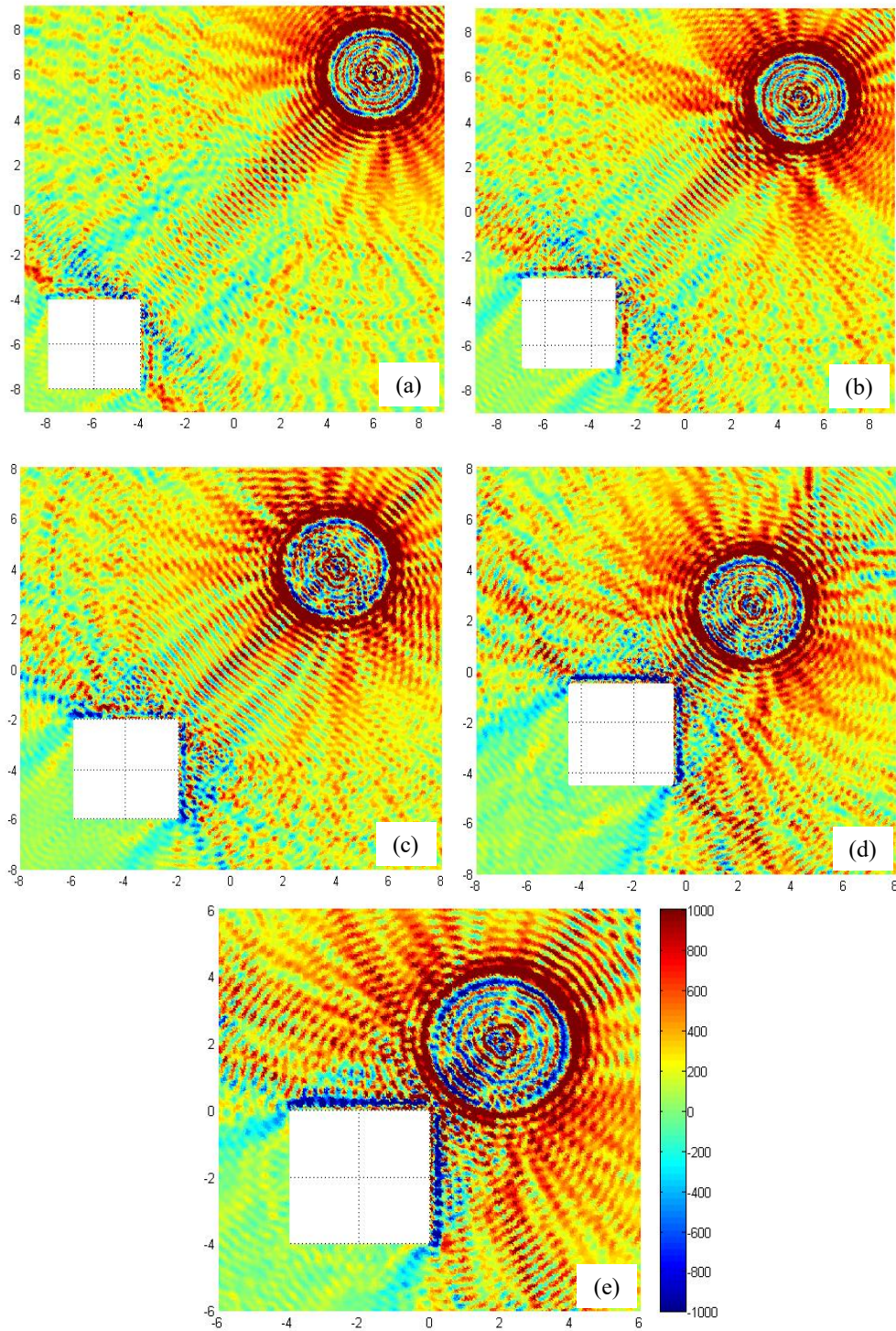


Figure 11: Identification of the square obstacle in the presence of a pre-existent circular obstacle. Contour plots for the topological derivative results as function of the distance between obstacles: (a) $d/r = 6\sqrt{2}$, (b) $d/r = 5\sqrt{2}$, (c) $d/r = 4\sqrt{2}$, (d) $d/r = 2.5\sqrt{2}$, (e) $d/r = 2\sqrt{2}$. The same color scale is used in all subfigures.

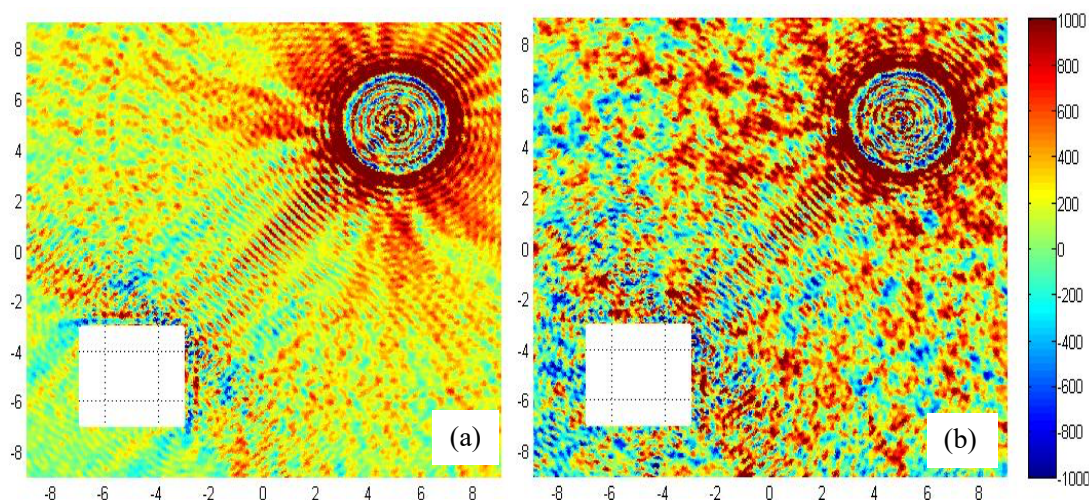


Figure 12: Assessment of the measuring error. Contour plots for the topological derivative results with (a) 10% and (b) 50% noise in the pressure values at the receptors.

mentation is proposed, which re-uses the matrices assembled for the solution of the forward problem for the computation of the adjoint problem.

Good quality results are obtained provided the lengths of the incident waves are smaller than one fifth of the characteristic size of the obstacles. An initial set-up with 50 sources and 100 receptors is recommended for practical identification problems. These settings can be adjusted later depending on the characteristics of each particular problem. The method shows low sensitivity to measurement error. Measurements with a noise of 50% the amplitude of the incident wave allow for positive identifications of the hidden obstacles. On the other hand, the flexibility of the method in the problem set-up makes it attractive for the implementation of adaptive strategies for the optimum placement of the sources and/or the receptors.

Although they have been practically shown for sound-hard obstacles, the formulations introduced for the inverse scattering problem and the BEM are general. They can handle problems involving penetrable obstacles (both, pre-existent and hidden ones), and even combinations of obstacles with different penetration coefficients. The solution of such problems will require the implementation of multi-domain BEM.

The extension of the analysis to three-dimensional problems leads to analogous formulae. See the works by Carpio and Rapún [2008b], Guzina and Bonnet [2006] and Nemitz and Bonnet [2008] for the details.

The method exhibits the potential to solve problems larger and more geometrically complex than those presented in this work. However, solving such problems would require faster algorithms to speed up not only the solution of limits but, more importantly, the post-processing on the BEM results at internal points for the computation of the D_T . In the BEM used in this work, the computational cost of solving direct and adjoint boundary problems is modest compared to post-processing at internal points.

The solution of the direct and adjoint boundary problems can be easily adapted to benefit from fast BEMs for acoustics, like the fast multipole in Nemitz and Bonnet [2008] and hierarchical matrices in Brancati et al. [2012]. On the other hand, iterative methods are an alternative to avoid the exhaustive sampling of the optimization for the computation of the D_T . A simply approach based on that introduced by Carpio and Rapún [2008a]. The idea is simple. Perform a first solution to the problem for a coarse array of internal points and use the points in which the D_T falls below a certain negative threshold to define a first guess the obstacle geometry. Then solve the problem for the new geometry and compute the T_D on a finer array of internal points. Such array of internal points only need to extend around the pre-existent obstacles and on zones with lowest values of the T_D in the previous step. Find the points with the lowest values of T_D for the new solution: those points which are close to any of the existing obstacles are included into that obstacle, whereas the points that far enough from the existing obstacles are used to create a new obstacle. Update the configuration and repeat the process until a stopping criterion (for instance, a threshold value of the D_T or a limit to the rate of change to the obstacle area) is fulfilled.

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