Chapter 9: Fundamental Concepts on Wavelet Transforms

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Fundamental Concepts on Wavelet Transforms

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Abstract

This chapter presents some fundamental concepts on wavelet transforms, both discrete and continuous, each are useful to understand the ability of the wavelet transform to capture sudden changes in material properties or parameters along the span of structural components being assessed. The chapter also discusses some families of wavelet transforms, some wavelet properties and regularization aspects.

Keywords: Wavelet Transform, Continuous Wavelet, Discrete Wavelet, Wavelet Family Function, Wavelet Packet.

1 Introduction of Wavelet Transform

Wavelet comes from the French word “ondalette”, which means small wave. Wavelets were first mentioned in the appendix of Haar's thesis (Haar, 1910). Haar wavelets remained anonymous for several years until in the 1930s (Reis, 2018). Working independently, various groups made research using the wavelet functions using a base and varying scales. On that occasion, using the Haar wavelets as a basis, Paul Levy investigated the Brownian motion (Reis, 2018). He showed that Haar-based functions were better than Fourier-based functions for studying the small Brownian motion with intricate details.

For a long period, Haar wavelets continued to be the only known orthonormal wavelet basis. In 1985, Mallat gave wavelets a big boost through his/her work in digital image processing. Meyer (1989), inspired by Mallat's results, constructed the first non-trivial (smooth) wavelet. Unlike the Haar wavelet, Meyer wavelet is continuously differentiable, but do not have compact support. In 1990, Ingrid Daubechies used Mallat's work to build a set of smooth wavelet orthonormal bases with compact supports. Daubechies' works are the foundation of current Wavelets applications (Palechor *et al.*, 2019; Silva *et al.*, 2019).

For a good understanding of wavelet analysis, it is necessary to start with the analysis of simple techniques. In practice, many signals can be represented in the time domain and in the frequency domain.

The representation of the signal in the frequency domain is obtained by applying the Fourier transform (FT) to the original signal expressed in the time domain. The result of this transformation is a set of frequencies that characterize the original signal. But the question arises: why do we need frequency information? Often the information that is needed cannot be seen in the time domain, but in the frequency domain. In other cases, the most important part of the signal information is “hidden” in their frequencies. This transformation can be applied to non-stationary signals; that is, signals that change their parameters over time.

In many applications, periodic oscillatory behavior is intermittent. For these analyses, the FT is no longer the most suitable as the series must be stationary (its parameters remain constant over time). As an alternative to solve this problem, there is the Short-Time Fourier Transform (STFT) (a generalization of the Windowed Gabor Transform) (Michel Misiti *et al.*, 1997). Its application allows the signal information achievement in time and frequency. The STFT methodology has some difficulties in determining the ideal time domain window width (e.g., interval of time that is slid along time series).

Wavelet analysis takes a different approach as it is based on the idea that any signal can be broken down into a series of basic functions called “wave”. This technique also allows the use of basic functions with variable size, and the use of long and short time and space intervals to capture the necessary information (Figure 1).

A great advantage provided by wavelets is the ability to perform local analysis. Consider a sinusoidal signal with a small break, so small that it is barely visible as in Figure 2.

Figure 1 – Wavelet Transform (adapted from Misiti et al. 1997).

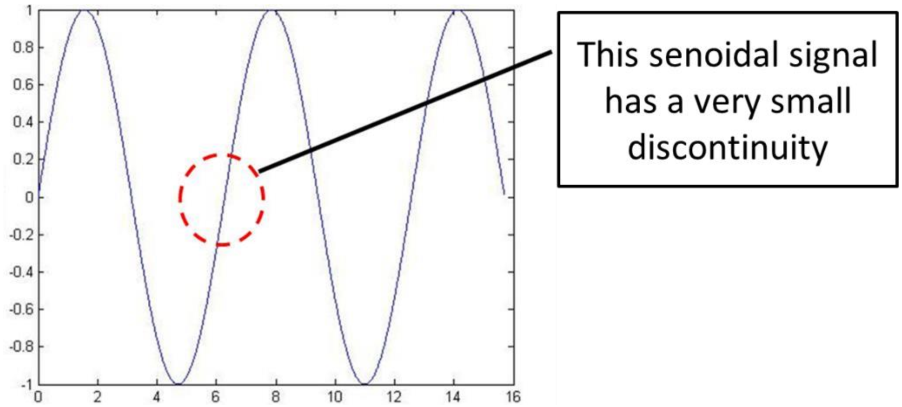
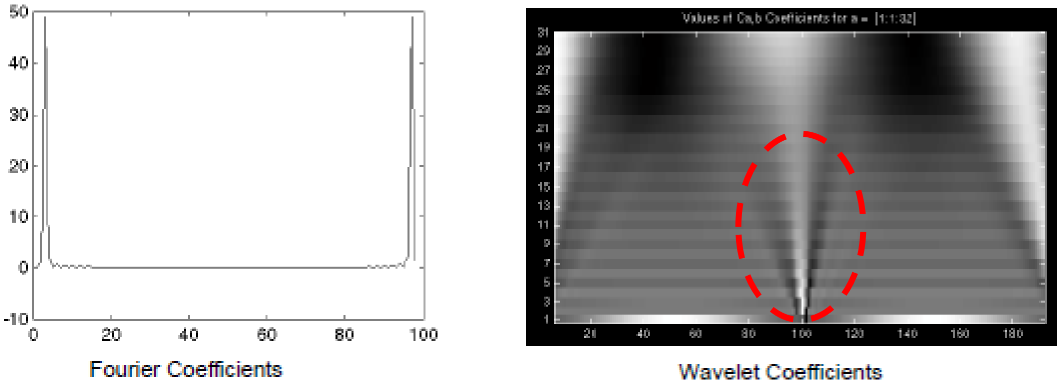


Figure 2 – Discontinuity on signal (adapted from M Misiti *et al.*, 1997)

A graph of Fourier amplitudes (Fourier coefficients) as a function of frequency, obtained by the FT of this sinusoidal signal, does not show anything, except the peaks due to the characteristic frequencies of sinusoidal signal, Figure 3(a). However, a portion of wavelet coefficients clearly shows the exact location in time of the discontinuity generated by the signal disturbance, Figure 3(b).



(a) (b)

Figure 3 – Comparation between Fourier coefficients (Weeks, 2007)**(a) and Wavelet coefficients** (M Misiti *et al.*, 1997) **(b)**

A wavelet is an effectively time-limited waveform with a zero-value averaging. One of the main advantages provided by the Wavelet Transform is the ability to perform local analysis; that is, it can analyze a restricted data series of a larger signal. This analysis can reveal aspects that other signal processing techniques cannot obtain, aspects such as: trends, points of degradation, discontinuities. In the case of damage identification, these discontinuities may be caused, for example, by cracks (Ovanesova, 2000; Ovanesova and Suárez, 2004).

The wavelet decomposition consists of calculating a "resemblance index" between the signal and the wavelet function. If the index is large then the similarity is strong, otherwise the similarity is weak. The Wavelet transform of a signal $\boldsymbol{f(x)}$, is the family $\boldsymbol{C(a,b)}$, which depends on two indices $\boldsymbol{a}$ and $\boldsymbol{b}$, the values $\boldsymbol{C(a,b)}$ are called coefficients, that will be described section 3.

Wavelets have been widely used to analyze time-domain signals. For the Wavelet analysis of spatial-domain signals, we can simply replace time with a spatial coordinate f(x), corresponding to vibration modes or displacements due to static load (Wu and Wang, 2011).

Similar to the windowed Fourier transform, the unidimensional Wavelet Transform projects a signal into two-dimensional space. The Wavelet Transform of the signal $f(x)$ is defined as:

$$W_{\psi}^{f}(a,b) = |a|^{-1/2} \int_{-\infty}^{\infty} f(x)\psi^{*}\left(\frac{x-b}{a}\right) dx \tag{1}$$

where $\psi^{*}(.)$ indicates the complex conjugate of $\psi(.)$ it is assumed that the mean value of the function $\psi(x)$ is null:

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 0 \tag{2}$$

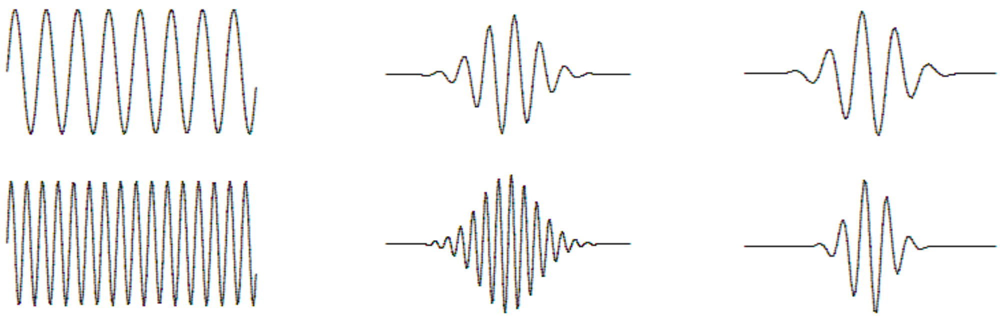
In both Short-Time Fourier Transform and the Wavelet Transform, the signal $f(x)$ is multiplied by a function of two variables. In the case of Short-Time Fourier transform variables, the function is as follows:

$$w^{w,\tau}(x) = \frac{1}{2\pi} w(x-\tau)e^{-iwx} \tag{3}$$

The respective function for the wavelet transform is given by:

$$\psi^{a,b}(x) = |a|^{-1/2}\psi^{*}\left(\frac{x-b}{a}\right) \tag{4}$$

The $\psi^{a,b}$ functions are called wavelets or mother wavelet functions. Short-time Fourier transform functions usually fluctuate and decay rapidly. In contrast to the $\psi^{a,b}(x)$ functions, the number of oscillations remains constant with window changes. This means that a wavelet is “scaled” along the axis of time (or space). For short-time Fourier transform, the size of the window remains constant while the number of oscillations changes. This principle is illustrated in Figure 4.



(a) (b) (c)

Figure 4 – Comparation between (a) Fourier transform, (b) short-time Fourier transform, (c) Wavelet transform (Michel Misiti *et al.*, 1997)**.**

For the wavelet transform analysis, $\psi(x)$ is the complex function of values located in the spatial domain x. The function $\psi(x)$ is the mother wavelet that generates the wavelet coefficients by shifting and scaling (Wu and Wang, 2011). The shifting from mother wavelet can be defined as:

$$\psi_{a,b}(x) = 2^{\frac{a}{2}}\psi(2^a x - b) \tag{5}$$

where a and b are scaling and shifting, respectively.

For a spatial signal $f(x)$ in the interval $[a, b]$, its wavelet transform is given by:

$$C_{a,b} = \int_{-\infty}^{\infty} f(x)\, \psi_{a,b}(x) dx \tag{6}$$

where $C_{a,b}$ is wavelet coefficient for mother wavelet $\psi_{a,b}\,(x)$ with scale a, and position b (Wu and Wang, 2011).

Wavelet Transform includes Continuous Wavelet Transform (TCW) and Discrete Wavelet Transform (TDW). The main advantage of TCW is its ability to provide time and scale information (Li *et al.*, 2009). The difference between the two transformations is in the form of scale representation (Table 1):

- In continuous analysis, the scale varies almost continuously between 2^1 and 2^5, for example. When a scale is small, only small details are analyzed. This is why continuous analysis is often easier to interpret (Ovanesova, 2000; Ovanesova and Suárez, 2004);
- In discrete analysis, the scale is dyadic, for example, 2^1, 2^2, 2^3, 2^4, and 2^5. Each level coefficient k is repeated 2^k times. This is why discrete analysis guarantees saving of coding space and is sufficient for synthesis (Ovanesova, 2000; Ovanesova and Suárez, 2004).

Table 1 – TCW and TDW differences (Ovanesova, 2000; Ovanesova and Suárez, 2004)

Continuous Time	Continuous Time	Discrete Time $(\Delta = 1)$
Continuous Analysis	**Continuous Analysis**	**Discrete Analysis**
$C_{a,b} = \int_R S(t)\frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)dt$ $a \in R^+, b \in R$	$C_{a,b} = \int_R S(t)\frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)dt$ $a = \Delta 2^j, b = \Delta k 2^j$ $j, k \in Z^2$	$C_{j,k} = \sum_{n \in Z} s(n) g_{j,k}(n)$ $a = 2^j, b = k2^j$ $j \in N, k \in Z$

2 Wavelets Properties

Wavelet functions have different properties suitable for certain purpose. According to Estrada (2008), the most relevant properties of the wavelet function to enable damage detection are:

PROPRIETY I – **Orthogonality and Biorthogonality:** Two functions $u(x)$ and $g(x)$ are orthogonal if their scalar product is null:

$$\langle u(x), g(x)\rangle = \int_a^b u(x)g^*(x)\,dx = 0 \tag{7}$$

where $g^*(x)$ is the complex conjugate of function $g(x)$. The term "biorthogonal" refers to two different bases orthogonal to each other, but the bases do not form an orthogonal set of functions.

These properties ensure fast determination of wavelet coefficients. Unfortunately, not all wavelet functions have the two following properties.

PROPRIETY II – **Compact Support:** support of a function is a set of points where the function is non-zero. A function has compact support if the adherence of the set, of non-null points, forms a closed and delimited set. This property means that the wavelet function does not assume a zero value for finite intervals, allowing for a more efficient representation of signals that have localized characteristics.

PROPRIETY III – **Vanishing Moment:** More precisely, if the mean value of $x^k\psi(x)$ is equal to zero, for $k = 0,1,\dots,n$, where $\psi(x)$ is the corresponding scaling wavelet function. Then the wavelet function has $n + 1$ vanishing moments and polynomials of degree n are suppressed by this wavelet function. This property determines the maximum degree of the polynomial that can be approximated. This property is used to select the most suitable mother wavelet function for damage detection.

PROPRIETY IV – **Regularity:** is the number of times a function is differentiable at point x_0. Singularities in a function can be detected by this property of regularity. $\boldsymbol{s}$ is the regularity of the function f; if the derivative of order m of f , at x_0, approaches $|x - x_0|^r$ locally around x_0, then $\boldsymbol{s} = m + r$, with $0 < r < 1$.

Table 2 – Proprieties of the mother wavelet functions.

Propriety	morl	mexh	meyr	haar	dbN	symN	coifN	BiorNr.Nd
Regular Infinitely	x	x	x					
Compact Orthogonal Support				x	x	x	x	
Compact Biorthogonal Support								x
Orthogonal			x	x	x	x	x	
Biorthogonal			x	x	x	x	x	x
Number of Arbitrary Null-Momentum					x	x	x	x
Continuous Transform	x	x	x	x	x	x	x	x
Discrete Transform			x	x	x	x	x	x

According to these properties, the most known mother wavelets are classified into (Ovanesova and Suárez, 2004):

- The wavelets functions Haar, Daubechies of n-th order, Meyer, Symlets of n-th order and Coiflets of n-th order are examples of orthogonal mother wavelets.
- The wavelets functions Haar, Daubechies of n-th order, Symlets of n-th order, and Coiflets are mother wavelets with compact support.
- The wavelets functions Daubechies of n-th order, Symlets of n-th order and Coiflets of n-th order are examples of mother wavelets with a number arbitrary of vanishing moments.
- The wavelets functions Morlet, Meyes and Gaussian are regular. Alternatively, the functions Daubechies of nth order, Symlets of n-th order and Coiflets of n-th order are mother wavelets with weak regularity.

3 Continuous Wavelets Transform (CWT)

The Fourier Transform is defined as the projection of the signal of the product of $f(t)$ and the exponential complex (orthogonal) function at time, as described in expression:

$$F(w) = \int_{-\infty}^{+\infty} f(t)e^{-jwt}dt \tag{8}$$

The TF results are Fourier coefficients, for each frequency ω. The reconstitution of the original signal $f(t)$ is obtained by multiplication of Fourier coefficients by harmonic function $\exp(\jmath\omega t)$ (Figure 5).

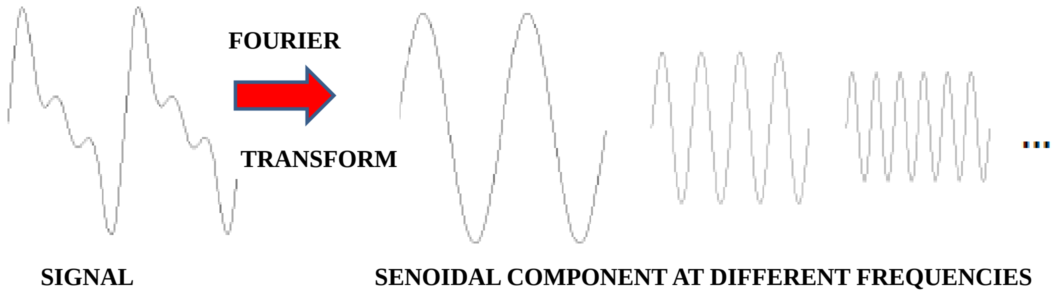


Figure 5 – Decomposition process of Fourier Transform (adapted from Misiti *et al.*, 1997)

Likewise, Continuous Wavelet Transform (CWT) is defined as the sum over entire time (or entire space) of the signal multiplied by mother wavelet function for a specific scale and position.

$$C(scale, position) = \int_{-\infty}^{+\infty} f(x)\psi(scale, position)dx \tag{9}$$

Results of CWT is the wavelet coefficient C. Such coefficients depend on a specific scale and position. Multiplying each wavelet coefficient by appropriately scaled mother wavelet produces the original signal (Figure 6).

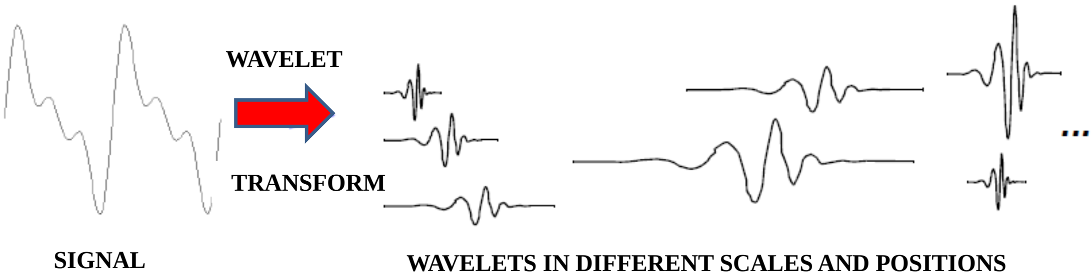


Figure 6 – Wavelet Transform Process (adapted from M Misiti *et al.*, 1997)**.**

3.1 Scale

Scaling a wavelet means stretching or compressing a function. Figure 7 present the effect of scale factor for a function$f(t)$, if the signal was a sinusoids.

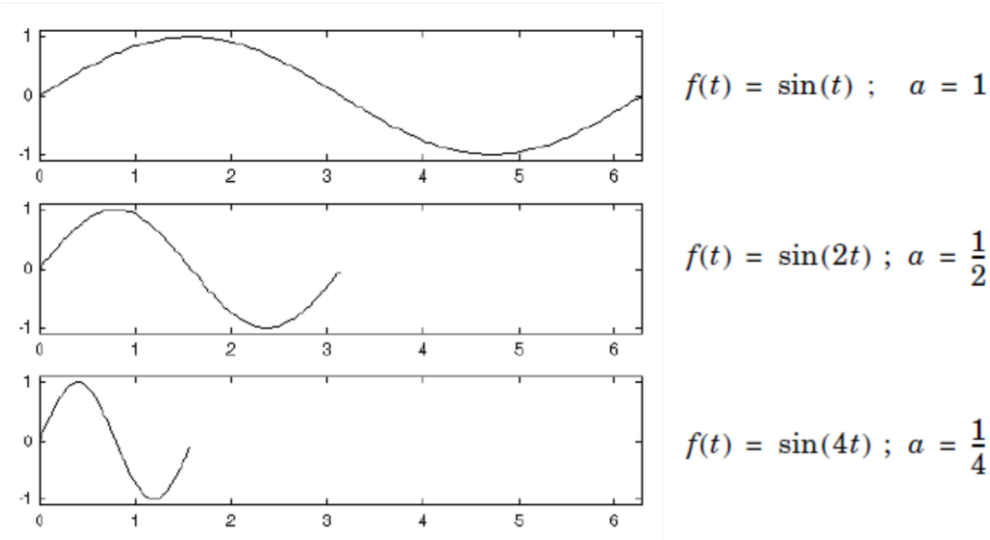


Figure 7 – Scale of Wavelet Transform (Weeks, 2007)**.**

3.2 Shifting

Shifting wavelets is simply delaying its onset along of signal. Mathematically, delaying $\boldsymbol{k}$ times, a function $\boldsymbol{f}(\boldsymbol{t})$ is represented by $\boldsymbol{f}(\boldsymbol{t}-\boldsymbol{k})$ (Figure 8).

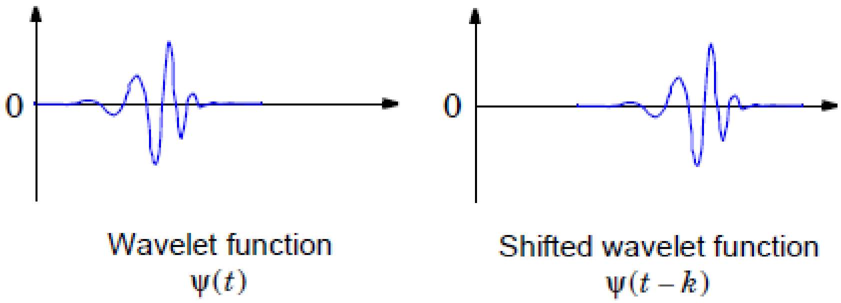


Figure 8- Wavelet function (Weeks, 2007)**.**

The signal CWT is the integration over signal multiplied by scale a and offset b. This process produces wavelet coefficients function of dimension a and position b. The option of CTW coefficients is performed by an algorithm in five steps:

1. Choose a mother wavelet and compare it to signal interval at the beginning of original signal.
2. Calculate a coefficient C representing the similarity ("resemblance index") between mother wavelet and original signal at analyzed interval. Note that the resulting coefficient depend on chosen wavelet shape (Figure 9).

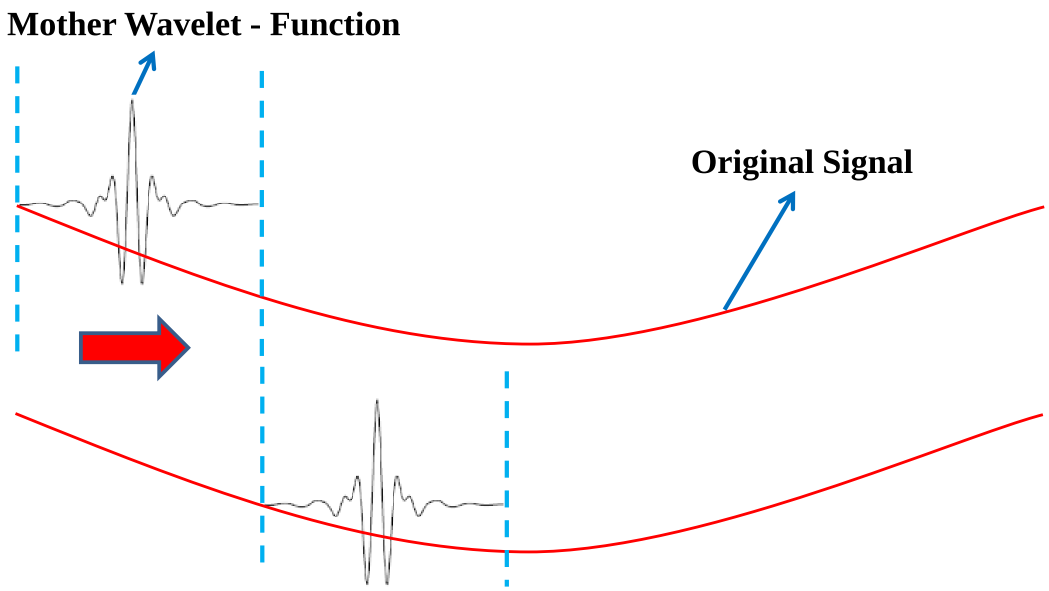


Figure 9 – Schematic calculus of wavelet coefficients (adapted from M Misiti *et al.*, 1997)**.**

3. Shift mother wavelet to the right and repeat steps 1 and 2 until it covered the entire signal (Figure 10).
4. Scale mother wavelet on analyzed stretch and repeat steps 1 to 3.

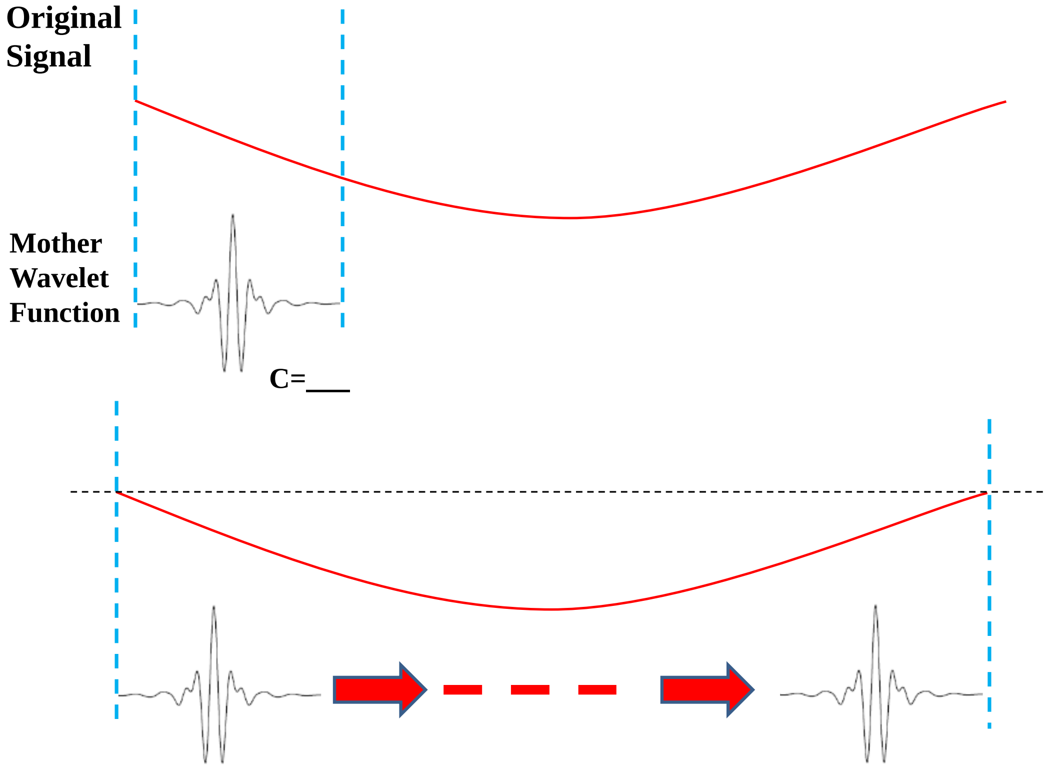


Figure 10 – Illustration of Wavelet scale (adapted from M Misiti *et al.*, 1997)**.**

5. Repeat steps 1 to 4 for all scales.

As a result, CWT produces wavelet coefficients for different scales. The x-axis represents position along the signal (time or space) and the y-axis represents the scale a. The color at each point (x, y) in space signal-scale represents the magnitude of wavelet coefficients C (Figure 11). Figure 12 shows the wavelet coefficients´ map generated by TCW.

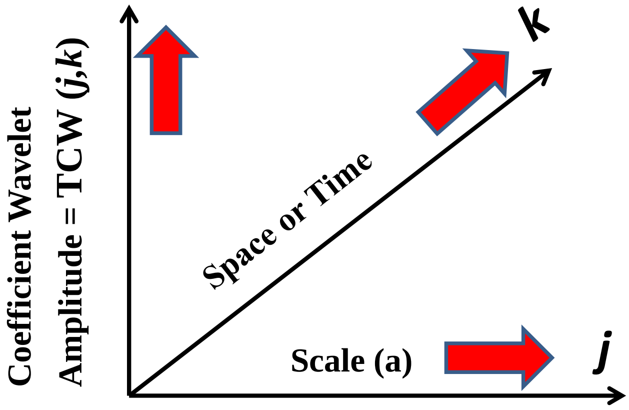
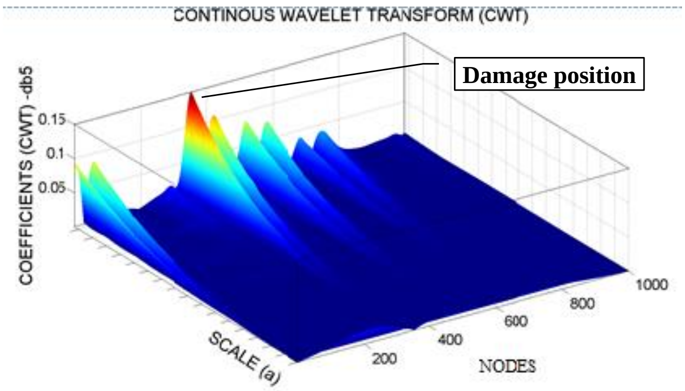
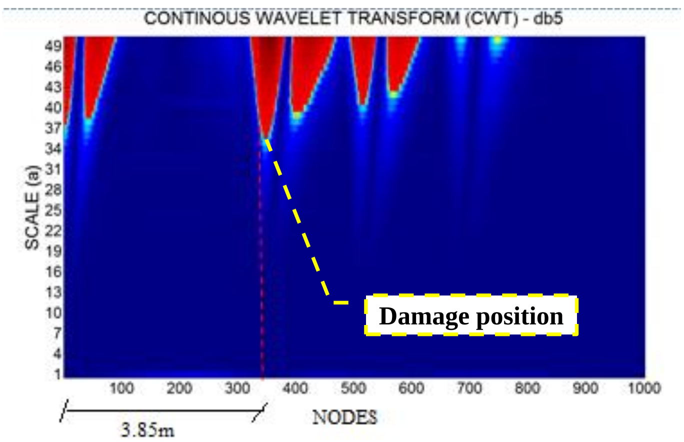


Figure 11 – Axis explanation of TCW graphs (modified by Gutierrez, 2002)**.**



(a) 3D



(b) 2D

Figure 12 – Examples of TCW graphs (Silva *et al.*, 2019)**.**

For CWT, the coefficient wavelet $\psi(a, b)$ surface can be described as an analytical function, function of parameters a (scale) and b (shifting) changing continuously over all space $\mathbb{R}^2$ (excluding $a = 0$). The CWT is defined by the following equation:

$$W_\psi^f(a, b) = |a|^{-1/2} \int_{-\infty}^{\infty} f(x)\psi\left(\frac{x-b}{a}\right) dx \tag{10}$$

3.3 CWT Analytical Example

Calculate the CWT of the following function

$$f(x) = e^{\frac{x^2}{2}} \tag{11}$$

using the Mexican hat wavelet (Ricker wavelet):

$$\begin{aligned} W_\psi^f(a, b) &= \frac{1}{\sqrt{a}} \int f(x)\psi^*\left(\frac{x-b}{a}\right) dx \\ &= \frac{1}{\sqrt{a}} \int e^{\frac{x^2}{2}} \left(1 - \left(\frac{x-b}{a}\right)^2\right) e^{-\frac{\left(\frac{x-b}{a}\right)^2}{2}} dx \end{aligned} \tag{12}$$

Scaling for $a = 1$ and shifting for $b = 0$, the coefficients can be obtained by the following expressions:

$$W(1,0) = \frac{1}{\sqrt{1}} \int e^{\frac{x^2}{2}} \left(1 - \left(\frac{x-0}{1}\right)^2\right) e^{-\frac{\left(\frac{x-0}{1}\right)^2}{2}} dx \tag{13}$$

In other terms,

$$\begin{aligned} W(1,0) &= \int e^{\frac{x^2}{2}}(1 - x^2)e^{-\frac{x^2}{2}} dx \\ &= \int (1 - x^2)e^0 \, dx \\ &= x - \frac{x^3}{3} + C \end{aligned} \tag{14}$$

The constant C appears once we have an indefinite integral. If we estimate $W(1,0)$ we compute $x = 10, -323{,}33$ plus the constant.

4 Discrete Wavelet Transform

The wavelet coefficients calculus on a continuum scale is a complicated task due to the generation of a fair amount of data. To minimize this task, only a discontinuous subset of scales and positions is chosen. This subset of scales and positions chosen are based on powers of 2 (dyadic scales) which is more efficiently computed. This approximate kind of wavelet analysis is called Discrete Wavelet Transform (DWT) (Ovanesova, 2000; Ovanesova and Suárez, 2004).

For this purpose, we define the scale $a = 2^j$ and the shifting $b = k(2^j)$, where $(j, k) \in Z$ and Z is integer set. Using these discrete parameters, DWT is given as:

$$TDW_{j,k} = 2^{-j/2} \int_{-\infty}^{\infty} f(x)\psi\left(2^{-j}x - k\right)dx = \int_{-\infty}^{\infty} f(x)\psi_{j,k}(x)dx \qquad (15)$$

The following three-step algorithm describes the stages for damage detection on a structure using DWT:

1. Obtain a signal or structure response associated with the complete structure or exam just a specific area of the structure.
2. Calculate the wavelet coefficients, performing signal DWT to different scales. Wavelet coefficients $C_{j,k}$ are obtained by:

$$C_{j,k} = \int_Z f(x)\,\psi_{j,k}(x)dx \qquad (16)$$

where the analyzed signal $f(x)$ is described by a scale $j \in N$ and position $k \in Z$; N and Z are the set of all positive scale integers and all position integers, respectively; and $\psi_{j,k}(x)$ is a mother wavelet and can be expressed by:

$$\psi_{j,k}(x) = 2^{0.5j}\,\psi_{j,k}(2^j x - k) \qquad (17)$$

3. Plot the wavelet coefficient for each scale level.

The examination of the distribution of the wavelet coefficients for each level is done so that suddenly changes can be observed (*i.e.*, a peak) meaning local disturbance. If a peak is not caused by a known source, such as geometric or material discontinuity, the detected disturbance means that there is damage near to disturbance peak location.

5 Wavelet Family

Mathematically, a function $\psi(x)$, to be considered a mother wavelet, if and only if, it belongs to $L^2(R)$ space (Daubechies, 1992; Mehra, 2018) and satisfy admissibility conditions. Without much mathematical rigor, a mother wavelet is a function that oscillates, has finite energy, and has an average value of zero. The different families of wavelet functions are enumerated below, presenting the wavelet families used to damage detection in next chapters. One note exhaustive list of wavelet families are presented in Mehra (2018).

5.1 Haar Wavelet Family

Haar wavelet is the first, and the simplest of all wavelets. Haar wavelet resembles a step function. It is a special case of Daubechies wavelet, the db1 wavelet itself.

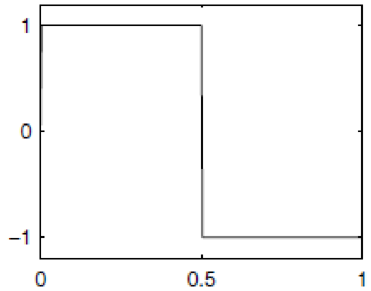


Figure 13- Harr Wavelet Family (Modified from M Misiti *et al.*, 1997)**.**

5.2 Daubechies Wavelet Family.

Ingrid Daubechies, one of the brightest stars in wavelet research world, invented the known orthonormal wavelets. The Daubechies wavelets names are also written as "dbN", where N is the order. As mentioned above, db1 wavelet is the Haar wavelet. Figure 14 show the next nine mother wavelet functions of Daubechies family.

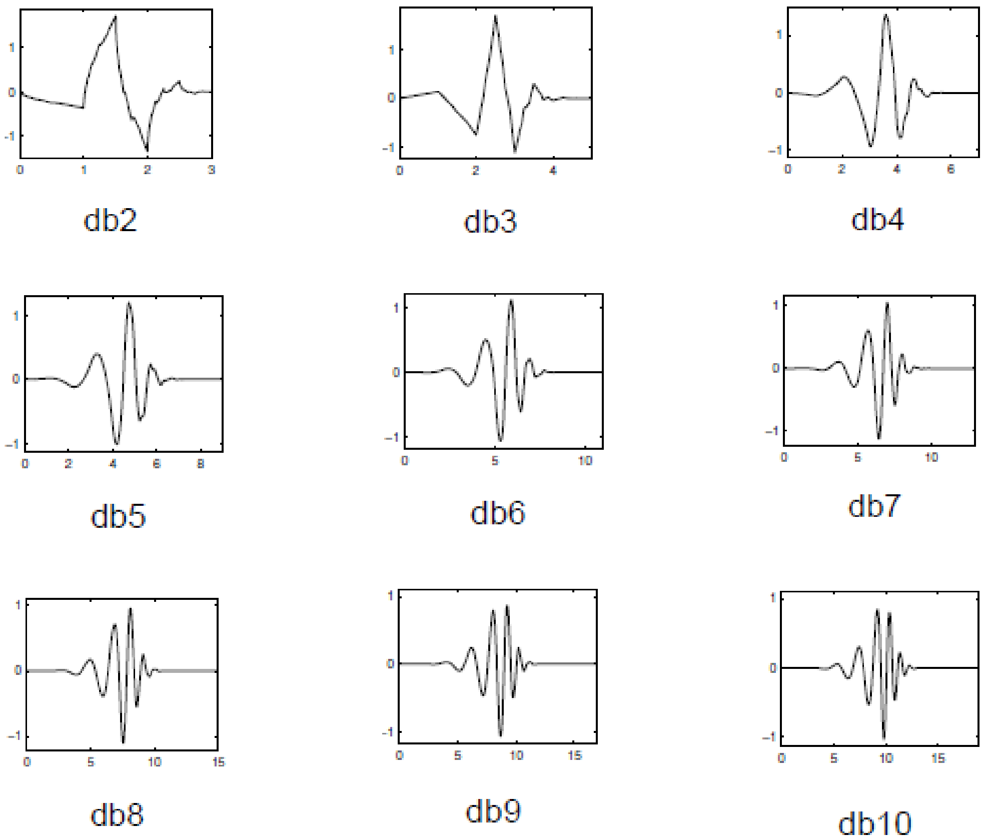


Figure 14- Daubechies Wavelets Family (Weeks, 2007)**.**

Orthogonal Daubechies wavelets, “dbN” are perfectly time compact. But in the frequency domain, they have a high degree of spectral superposition between the scales. The orthogonality is their main advantage. An error in the input signal does not increase after the transformation. This property ensures computational and numerical stability.

5.2.1 Coiflets Wavelet Family

Concepted by Daubechies by Coffman's request, the mother wavelet has $2N$ and scale function has $2N-1$ vanishing moments, respectively. The two functions have a support length of $6N-1$ (Figure 15).

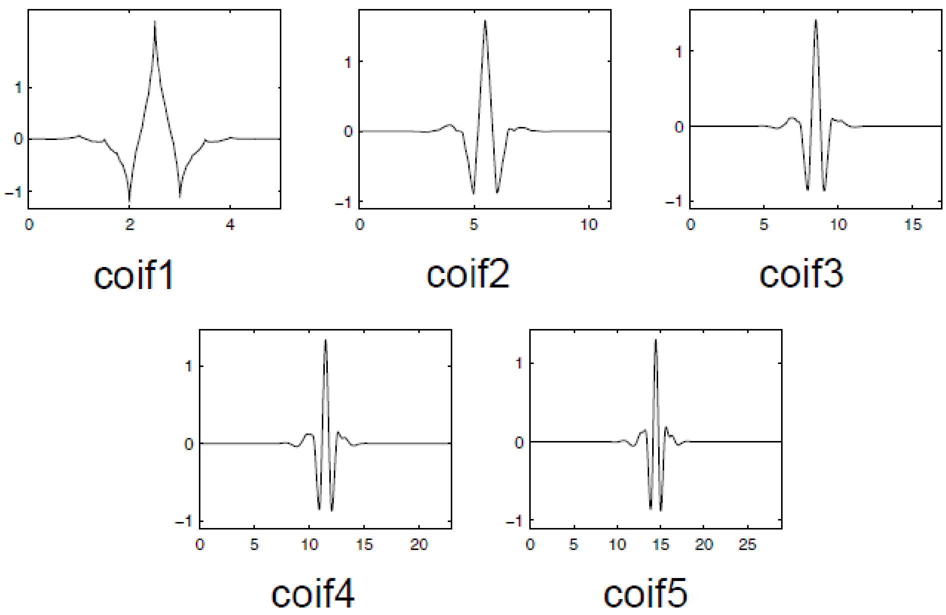


Figure 15- Coiflet Wavelets Family (Daubechies, 1992)**.**

5.3 Biorthogonal Wavelet Family

Biorthogonal wavelet bases (Daubechies, 1992) were conceived to obtain a symmetric wavelets family with compact support (De Souza *et al.*, 2007). Figure 16 shows some examples of biorthogonal wavelets.

5.4 Symlets Wavelet Family.

Proposed as dbN family modification by Daubechies, Symlet wavelets are almost symmetrical. The properties of dbN and Sym family are similar. But Symlet functions tend to be symmetric. Figure 17 presents examples of Symlet wavelet functions.

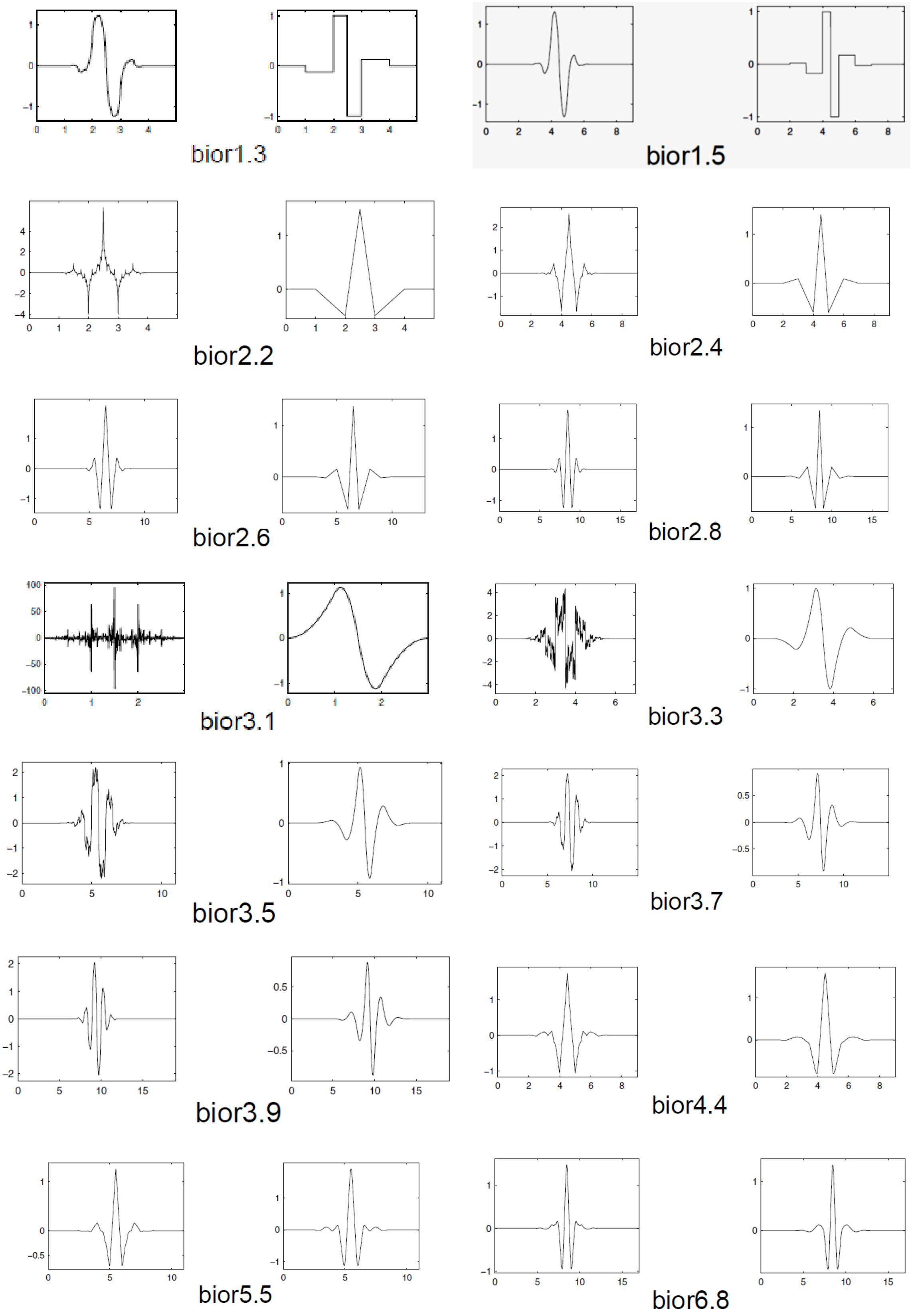


Figure 16- Daubechies Wavelets Family (Daubechies, 1992)**.**

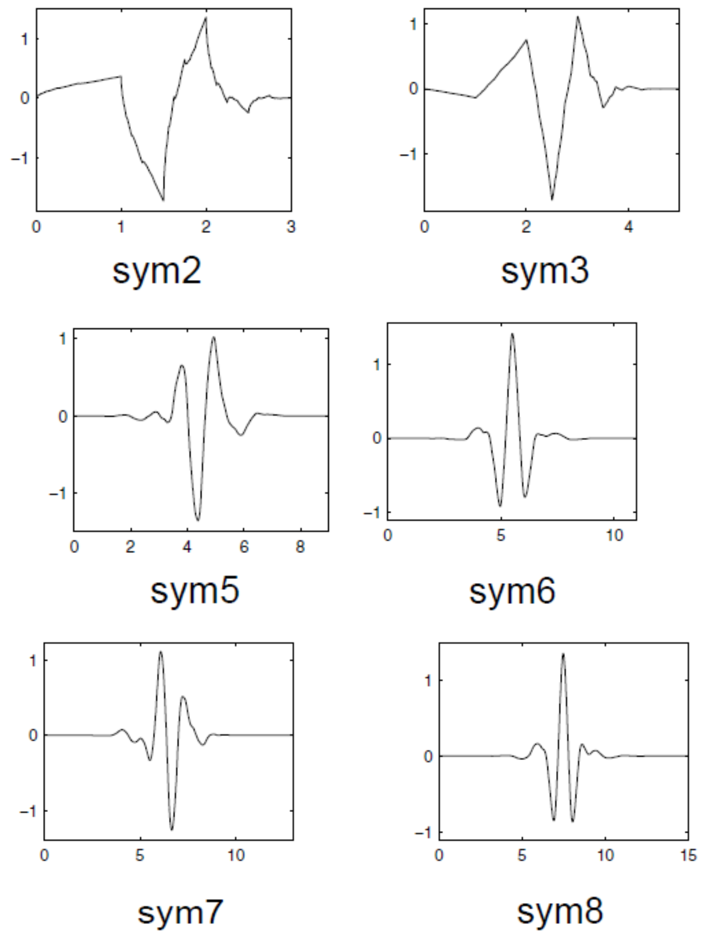


Figure 17- Symlet Wavelets Family.

5.5 Recommendations to choose a Wavelet family

One of the main criticisms directed towards the wavelet transform is the criteria to choose the better mother wavelet function $\psi^{a,b}$. To aid in this choice, there are a series of criteria and recommendations in the literature to be considered. After choosing a wavelet family, Torrence and Compo (1998) enumerate some important considerations, as presented below:

- ✓ **Orthogonal or non-orthogonal criteria**
  The wavelet transform, using families of orthogonal wavelets (Meyer, 1989), provides a more compact representation of the analyzed signal. Conversely, the Wavelet transform obtained by non-orthogonal wavelet families is highly redundant at larger scales, in which the Wavelet Spectrum at adjacent times is highly correlated (Meyer, 1989). The non-orthogonal Wavelet Transform is useful in analyzing time series (also valid for spatial series) where smooth and continuous variations in amplitude are expected.

- ✓ **Complex or real criteria**:
  A complex wavelet function, providing amplitude and phase information, is better suited to capture time-series oscillatory behavior. Instead, a real wavelet

function only provides information about component amplitude. This wavelet function can only be used to locate peaks and discontinuities.

- ✓ **Criteria support:**
  The wavelet function resolution is determined by the balance between its support in real space and frequency space. A function with more compact (narrower) support will have good time-domain resolution and poorer frequency-domain resolution. While a function with wider (broader) support will have a poorer resolution in time-domain resolution, and a good resolution in frequency-domain (a consequent characteristic which is due to the Heisenberg uncertainty principle).

- ✓ **Format Criteria:**
  The wavelet function chosen must reflect the characteristics of the analyzed signal. For series with peaks or discontinuities, a good choice would be Haar wavelet. While, for smoother series with more subtle variations, a wavelet function such Morlet wavelet family should be chosen. If the main interest is to obtain the Wavelet Energy Spectrum, then the choice of the wavelet function is not critical and any one of them will provide the same qualitative result.

6 Wavelet Packet Transform

The Wavelet Packet Transform (WPT) is a technique that decompose repeatedly a signal into successive low and high frequency components using a recursive decimation filter operation. This technique was first introduced by Coifman, Meyer and Wickerhauser (Mallat, 1999; Peng, Hao and Li, 2012). Wavelet packages consist of a usual linearly combined wavelet functions family. Figure 18 shows binary tree of a temporal signal $f(t)$ up to the 3rd level of WPT.

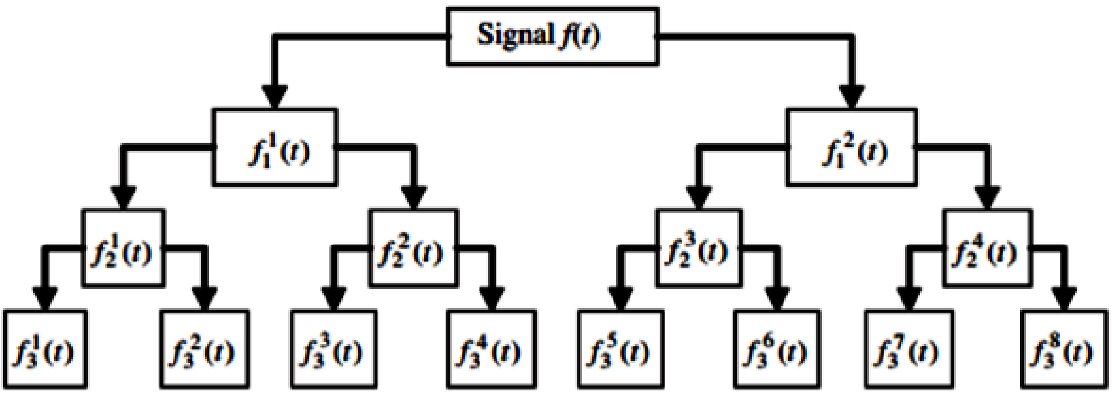


Figure 18- Wavelet Packet Decomposition (WPD) (adapted from Peng, Hao and Li, 2012).

After the j-th level of decomposition, the original signal $f(t)$ can be constructed by the sum of $2j$ components as follows:

$$f(t) = \sum_{i=1}^{2j} f_j^i(t) \tag{18}$$

where, $f_j^i(t)$ is the Wavelet Packet signal component that can be expressed by a linear combination of wavelet functions:

$$f_j^i(t) = \sum_{k=-\infty}^{\infty} c_{j,k}^i\,(t)\psi_{j,k}^i(t) \tag{19}$$

where the indexes i, j and k are integers defined as modulation, scale and shifting parameter, respectively. The coefficient $c_{j,k}^i$ and the function $\psi_{i,k}^i(t)$ are, respectively, Wavelet Packet coefficient and function. The Wavelet Packet coefficient $c_{j,k}^i$ is obtained by orthogonal projection of the signal $f(t)$ by report to Wavelet Packet family $\psi_{j,k}^j(t)$, as described as follows:

$$c_{j,k}^i \;= \int_{-\infty}^{\infty} f(t)\psi_{j,k}^i(t)dt \tag{20}$$

And Wavelet Packet function is defined as:

$$\psi_{j,k}^i(x) = 2^{j/2}\psi^i\left(2^j t - k\right) \tag{21}$$

where $\psi^1(t) = \psi \quad (t)$ is defined as mother wavelet function. And wavelet functions ψ^i $(i \;\geq\; 1)$ are function of recursive relationships:

$$\psi^{2i} = \sqrt{2} \sum_{k=-\infty}^{\infty} h(k)\,\psi^i(2t - k) \tag{22}$$

$$\psi^{2i+1} = \sqrt{2} \sum_{k=-\infty}^{\infty} g(k)\,\psi^i(2t - k) \tag{23}$$

where $h(k)$ and $g(k)$ are Quadrature Mirror Filters (QMF) associated to scale function and the mother wavelet function.

Each component in the Wavelet Packet Decomposition (WPD) tree can be seen as output from a filter tuned to a particular base function (Figure 18). At WPD top (root node), where decomposition level is low, we have a good time-domain resolution but a poor frequency-domain resolution. At WPD bottom (leaf node), where decomposition level is relatively high, we have a good frequency-domain resolution but a poor time-domain resolution. For structural integrity monitoring purposes, good frequency domain resolution is more important and thus, a high level of DWP is often needed to detect minor changes in signals (Sun and Chang, 2002, 2004).

6.1 Wavelet Packet Energy

Wavelet Packet Energy has demonstrated to be a more robust tool for damage identification compared to Wavelet Packet coefficients (Sun and Chang, 2002; Peng, Hao and Li, 2012).

The energy of the wavelet packet signal is given by:

$$E_f = \int_{-\infty}^{\infty} f^2(t)dt = \sum_{m=1}^{2^j}\sum_{n=1}^{2^j}\int_{-\infty}^{\infty} f_j^{\ m}(t) f_j^{\ n}(t)dt \tag{24}$$

where $f_j^{\ m}$ and $f_j^{\ n}$ are decomposed wavelet components. The total energy of the signal is expressed as the energy summation of wavelet packet components when mother wavelet is orthogonal:

$$E_f = \sum_{n=1}^{2^j} E_{f_j^i} = \sum_{i=1}^{2^j}\int_{-\infty}^{\infty} f_j^{\ i}(t)^2\,(t)dt \tag{25}$$

From Eqs. (24) and (25), the signal component $f_j^{\ i}(t)$ is a superposition of wavelet functions $\psi_{j,k}^{i}\,(t)$ of the same scale j, but translated in the time domain $(-\infty < k < +\infty)$. The energy component $E_{f_j^i}$ is the energy stored in a frequency band determined by wavelet functions $\psi_{j,k}^{i}(t)$.

7 Regularization Methods

The Wavelets coefficient disturbance is generated by several causes (e.g., signal noisy). To reduce this disturbance, Tikhonov regularization method can be applied (Tikhonov and Arsenin, 1977).

In general, inverse problems are ill-posed and their solutions are very sensible to noise. Small errors, due to experimental measured data, can result in a significant difference in ill-posed problems. Regularization methods seek to reduce oscillations in numerical solution by modification of objective function (Tikhonov and Arsenin, 1977; Schnur and Zabaras, 1990; Bezerra and Saigal, 1993; Bezerra, 1995; Tikhonov *et al.*, 1995; Silva Neto and Moura Neto, 2005). The most used “regularization” terms are zero-order, first-order, and second-order terms (Beck, Blackwell and Clair Jr, 1985; Silva Neto and Moura Neto, 2005). The zero-order term controls change in the magnitude of vector u, the first-order term controls change in the amplitude of the rate of change of vector u, and second-order terms can be expressed in integral form as (Schnur and Zabaras, 1990):

$$\rho = \beta_0 \int (u^2)\,\mathrm{ds} + \beta_1 \int \left(\frac{\partial u}{\partial s}\right)^2 \mathrm{ds} + \beta_2 \int \left(\frac{\partial^2 u}{\partial s^2}\right)^2 \mathrm{d}s \tag{26}$$

In finite differences, an analogous regularization equation is written as:

$$\rho = \beta_0 \sum_{\mathrm{i}=1}^{\mathrm{p}} \left(u_{\mathrm{i}}^{(n)}\right)^2 + \beta_1 \sum_{\mathrm{i}=1}^{\mathrm{p}} \left(u_{\mathrm{i}}^{(n)} - u_i^{(n-1)}\right)^2 + \beta_2 \sum_{\mathrm{i}=1}^{\mathrm{p}} \left(u_i^{(n)} - 2u_{\mathrm{i}}^{(n-1)} + u_{\mathrm{i}}^{(n-2)}\right)^2 \tag{27}$$

where, to an iteration number n, β_j is regularization parameters, s spatial parameter, and u_i the components of u. According to Beck *et al* (1985), Equation (27) is analogous to Equation (26). The finite difference regularization expression (27) is simple to implement.

With large values of β_j, variations of the vector u are obtained and tend to delay convergence. While small values of β_j can result in large oscillations of solution (Bezerra and Saigal, 1993; Bezerra, 1994, 1995).

EXAMPLE 01: Wavelet Signal Regularization

To increase the changes caused by damage at response signal, Tikhonov regularization method was applied. Figure 19 shows difference between the wavelet coefficients for a smoothed and an unregulated signal.

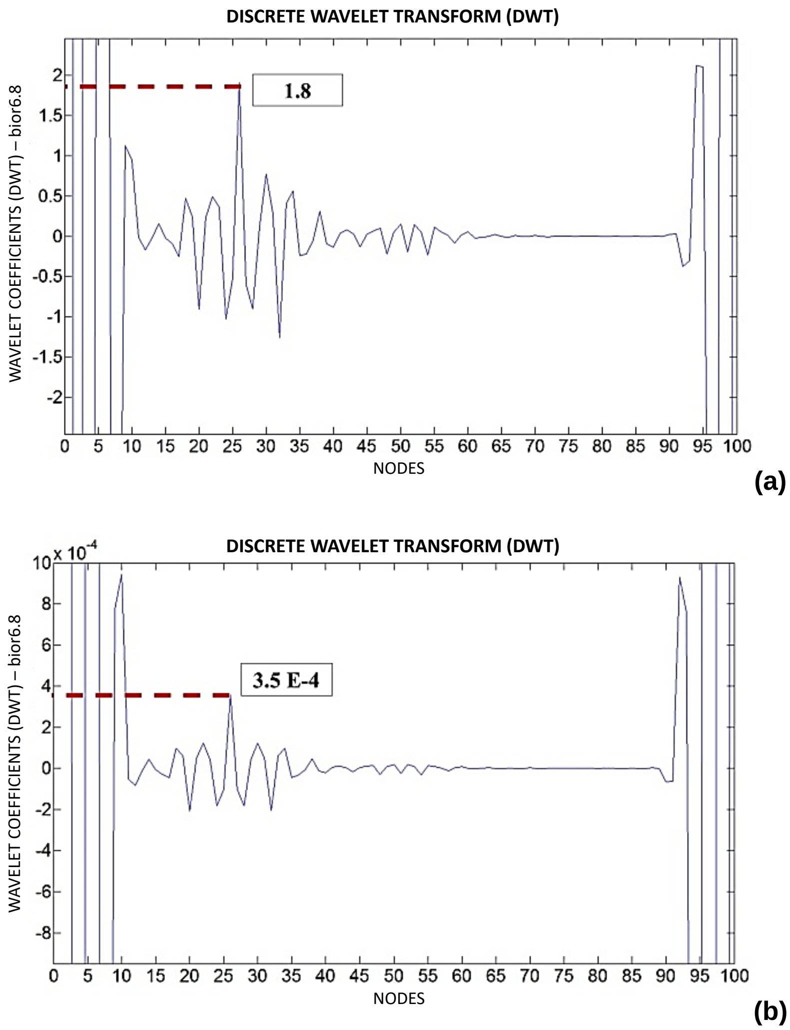


Figure 19 – Wavelet coefficients (a) with and (b) without Tikhonov Regularization.

In the Annex, Algorithm 3 is the Tikhonov Regularization coded in MATLAB.

8 Concluding Remarks

In this chapter some fundamental concepts on wavelet transforms were presented, both for discrete and continuous cases. The wavelet transforms are useful to understand the ability of wavelet transform to capture sudden changes in material properties or parameters along the span of structural component being assessed. Several families of wavelet transforms were discussed, including some wavelet properties and regularization aspects.

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Annex

Algorithm 1 – Signal Discontinuity

Algorithm 1 – Signal discontinuity (MATLAB implementation).

```
t1= 0:0.1:6; x1= sin(2*pi*t1);
t2= 6:0.1:6; x2= sin(2*pi*t2+0.1);
t = [t1 t2]; x = [x1 x2]; plot(t,x)
```

Algorithm 2 – Cubic spline interpolation

Algorithm 2 – Cubic spline interpolation (MATLAB implementation).

```
x = [1 2 3 4 5];
y = [0 1 0 1 0];
xx = 1:.05:5;
yy = spline(x,y,xx);
x_interp = 1.5;
y_interp = spline(x,y,x_interp)
y_interp = 1.1250
plot(x,y,'o',xx,yy,'r-',x_interp,y_interp,'*')
```

Algorithm 3 – Tikhonov Regularization

Algorithm 3 – Tikhonov regularization (MATLAB implementation).

```
B0=100;
B1=100;
B2=100;
```

```
n=length(u);
Part1=0;
Part2=0;
Part3=0;
for i=1:n
     if i==1
          a=(u(i,2))^2;
          Part1=a;
          b=((u(i,2)))^2;
          Part2=b;
          c=((u(i,2)))^2;
          Part3=c;
          u_reg(i)=(B0*(Part1))+(B1*(Part2))+(B2*(Part3));
      elseif i==2
          a=(u(i,2))^2;
          Part1=a;
          b=((u(i,2))-(u(i-1,2)))^2;
          Part2=b;
          c=((u(i,2))-(2*(u(i-1,2))))^2;
          Part3=c;
          u_reg(i)=(B0*(Part1))+(B1*(Part2))+(B2*(Part3));
      else
          a=(u(i,2))^2;
          Part1=a;
          b=((u(i,2))-(u(i-1,2)))^2;
          Part2=b;
          c=((u(i,2))-(2*(u(i-1,2)))+u(i-2,2))^2;
          Part3=c;
          u_reg(i)=(B0*(Part1))+(B1*(Part2))+(B2*(Part3));
      end
end
figure
hold on
grid on
plot(u(:,1),u(:,2),'b')
plot(u(:,1), u_reg(:),'r')
title('TIKHONOV Regulatization')
```

where, `u_reg(i)` is the regularization vector of signal `u(:,2)`.