Chapter 18

An Application of Finite Element Method and Sensitivity Analysis in Structural Dynamics

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An Application of Finite Element Method and Sensitivity Analysis in Structural Dynamics

Suzana Moreira Avila[1*] and Zenon J. Guzman del Prado[2],

[1]Post-Graduate Program - Integrity of Engineering Materials, University of Brasilia, Brazil. E-mail: avilas@unb.br
[2]Graduate Program in Geotechnics, Structures and Civil Construction, Federal University of Goiás, Brazil. E-mail: zenon@ufg.br

*Corresponding author

Abstract

In this chapter a short description of finite element method applied to linear structural dynamics is initially described. Modal analysis, modal superposition method, and numerical methods for the eigenvalue and eigenvector problem are described. Finally, a sensitivity analysis of free vibration problem applied to plane frames is performed to study the sensitivity of loss of stiffness on the natural frequencies and corresponding vibration modes.

Keywords: Structural dynamics, finite element method, sensitivity analysis, eigenvalue problem

1 Introduction

Structural dynamics is a research area focused on the analysis of the behavior of structural members subjected to dynamic loads, responsible for generating a response that includes the emergence of accelerations and velocities, in addition to displacements, all varying in time. Dynamic loads originating from several factors (people, wind, sea waves, vehicle traffic, earthquakes and explosions, among others) are characterized by having their magnitude, direction and application point varying in time. Applications of structural dynamics can be widely found in several engineering areas, such as: aerospace, automotive, mechanical and civil engineering. It is worth mentioning that the principles and solution techniques are the same, despite the variety of applications (Figure 1).

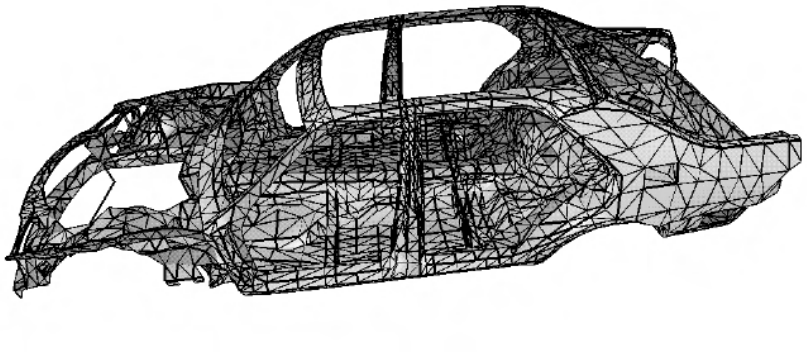


Figure 1 – Finite Element Model of the Car Structure (Hemez & Pagnacco [2000])

The importance of this kind of analysis is supported by the occurrence of structural failures caused by excessive dynamic response, which can result not only in impairing the functioning of machines and equipment, in economic damage due to the structural collapse, but also in the inestimable loss of human lives.

Structural dynamics has a vast literature available and several good textbooks. (Paz & Kim [2019], Cook et al [1989]; Clough & Penzien [1993], Chopra [1995], Craig & Kurdila [2006], Tedesco et al [1998], Rao [2011]). The problems studied in this area include, among others, two broad categories: obtaining frequencies and associated vibration modes and analyzing the response in the time domain.

The Finite Element Method (FEM) was originally conceived by Clough in the 60s (Clough [1960]), currently it is a computational tool consecrated not only in structural analysis, but also in solving problems in the areas of fluid dynamics and thermodynamics, among others. It is a numerical procedure to solve engineering boundary value problems that are very complex to solve via classical analytical methods. This complexity is present for example on non trivial geometrical configurations, material properties or loading conditions. The philosophy of the method is based on discretizing the continuum into a finite number of elements interconnected at certain points called nodes or joints, building a system mesh. Formulating the boundary value problem results in a system of algebraic equations. The idea is that the solution of the various appropriately selected elements falls into a solution that converges to the exact solution of the global system as the size of the elements is reduced.

The application of FEM to the structural dynamic analysis is made by establishing the equations of motion for a finite element given, then setting up the global equations of motion of the system from the chosen discretized mesh. The problem lies in the solution of the resulting system of ordinary differential equations for the case of undamped free vibrations that provide natural frequencies and associated modes of vibration. Additionally, the solution of the forced vibration problem will result in the response of the structure in the time domain.

In this chapter, a short description of finite element modelling will be described, then modal analysis and orthogonality of modes will be considered. Description of different methods to solve the eigenvalue and eigenvector problem is considered, as well as modal superposition

method. Finally, sensitivity analysis is performed to study the effect of loss of stiffness in the system natural frequencies and associated mode vibrations.

2 Modelling in Finite Element

To perform a dynamic analysis using the finite element method, the system of dynamic equilibrium equations should be assembled. For this, the element time dependent field displacements vector $\hat{\mathbf{u}}(t, x_1, x_2, x_3)_{(e)}$, in x_1, x_2, and x_3 directions, is composed by displacements $u_1(t,x_1,x_2,x_3)$, $u_2(t,x_1,x_2,x_3)$ and $u_3(t,x_1,x_2,x_3)$, respectively. This displacements vector can be described as a linear combination of time dependent nodal displacements $\mathbf{u}(t)$ and shape functions $\mathbf{N}(x_1,x_2,x_3)$ given by (Busby and Staab [2018], Bathe [1996] and Rao [2004]):

$$\hat{\mathbf{u}}(t, x_1, x_2, x_3)_{(e)} = \begin{Bmatrix} u_1(t, x_1, x_2, x_3) \\ u_2(t, x_1, x_2, x_3) \\ u_3(t, x_1, x_2, x_3) \end{Bmatrix}_{(e)} = \mathbf{N}(x_1, x_2, x_3)\mathbf{u}(t) \tag{1}$$

Considering both small displacements and small rotations and linear elastic material, the strain-displacements, and stress-strain relations, in index notation, are given by:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \quad i,j = 1,2,3 \tag{2}$$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \qquad i,j,k,l = 1,2,3 \tag{3}$$

where ε_{ij} are the linear strain components, σ_{ij} are the stress components and C_{ijkl} is the fourth order constitutive tensor.

Substituting Eq (1) into Eq. (2), the finite element strain-stress relations can be given as:

$$\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{N}(x_1, x_2, x_3)\mathbf{u}(t)_e = \mathbf{B}\mathbf{u}(t)_e \tag{4}$$

where **D** is the derivatives matrix and **B** is the deformation matrix.

Substituting Eq. (4) in Eq. (3) the stress-strain relations result as:

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon} = \mathbf{C}\mathbf{B}\mathbf{u}(t)_e \tag{5}$$

where **C** is the constitutive matrix.

The elastic strain energy of a finite element can be written as:

$$\mathrm{U}^{(e)} = \frac{1}{2}\iiint_{V^{(e)}} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dV \tag{6}$$

The total work of external forces is given by:

$$\mathrm{W}^{(e)} = \iiint_{V^{(e)}} \hat{\mathbf{u}}_{(e)}^T \mathbf{b}\; dV + \iint_{S_t^{(e)}} \hat{\mathbf{u}}_{(e)}^T \mathbf{t}\, dS + \sum_{i=1}^{N} \hat{\mathbf{u}}_{(e)}^T \mathbf{F}_i \tag{7}$$

Where vector **b** represents the body forces of the finite element, vector **t** represents the surface forces acting on finite element and $\mathbf{F}_i$ represents the concentrated nodal forces acting on the finite element.

Then, the potential energy of a finite element can be written as:

$$\Pi^{(e)} = U - W$$

$$\Pi^{(e)} = \frac{1}{2}\iiint_{V^{(e)}} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dV - \iiint_{V^{(e)}} \hat{\mathbf{u}}_{(e)}^T \mathbf{b}\ dV - \iint_{S_t^{(e)}} \hat{\mathbf{u}}_{(e)}^T \mathbf{t}\, dS - \sum_{i=1}^{N} \hat{\mathbf{u}}_{(e)}^T \mathbf{F}_i \tag{8}$$

Substituting Eq. (4) and Eq. (5) in Eq. (8), the potential energy can be written as:

$$\Pi^{(e)} = \frac{1}{2}\mathbf{u}^{\mathrm{T}} \iiint_{V(e)} \mathbf{B}^{\mathrm{T}}\mathbf{C}\mathbf{B}\, dV\, \mathbf{u} - \mathbf{u}^{\mathrm{T}}\left(\iiint_{V^{(e)}} \mathbf{N}^{\mathrm{T}}\mathbf{b}\, dV - \iint_{S_t^{(e)}} \mathbf{N}^{\mathrm{T}}\mathbf{t} dS - \sum_{i=1}^{N} \mathbf{N}^{\mathrm{T}}\mathbf{F}_i\right) \tag{9}$$

and simplifying:

$$\Pi^{(e)} = \frac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{k}\mathbf{u} - \mathbf{u}^{\mathrm{T}}\mathbf{f} \tag{10}$$

Where the finite element stiffness matrix is given by:

$$\mathbf{k} = \iiint_{V(e)} \mathbf{B}^{\mathrm{T}}\mathbf{C}\mathbf{B}\, dV \tag{11}$$

And the finite element load vector is given by:

$$\mathbf{f} = \iiint_{V^{(e)}} \mathbf{N}^{\mathrm{T}}\mathbf{b}\, dV + \iint_{S_t^{(e)}} \mathbf{N}^{\mathrm{T}}\mathbf{t} dS + \sum_{i=1}^{N} \mathbf{N}^{\mathrm{T}}\mathbf{F}_i \tag{12}$$

For a finite element, the kinetic energy can be written as:

$$T^{(e)} = \frac{1}{2}\iiint_{V^{(e)}} \rho \dot{\hat{\mathbf{u}}}_{(e)}^T \dot{\hat{\mathbf{u}}}_{(e)}\, dV \tag{13}$$

Substituting Eq. (1) in Eq. (13), the kinetic energy of a finite element results in:

$$T^{(e)} = \frac{1}{2}\iiint_{V^{(e)}} \rho \dot{\mathbf{u}}^T \mathbf{N}^T \mathbf{N} \dot{\mathbf{u}}\, dV = \frac{1}{2}\dot{\mathbf{u}}^T \left[\iiint_{V^{(e)}} \rho \mathbf{N}^T \mathbf{N}\, dV\right] \dot{\mathbf{u}} = \frac{1}{2}\dot{\mathbf{u}}^T \mathbf{m} \dot{\mathbf{u}} \tag{14}$$

Where m is the mass matrix of the finite element given by

$$\mathbf{m} = \iiint_{V^{(e)}} \rho \mathbf{N}^T \mathbf{N}\, dV \tag{15}$$

Matrix **m** is known as consistent matrix because it is a full matrix containing the coupling of inertial contribution of all nodal displacements. However, it is possible to consider a

lumped mass matrix where there is not coupling between inertial contribution of nodal displacements, and it is a diagonal mass matrix and every term in diagonal is associated to a concentrated mass in the direction of each nodal displacement.

Energy dissipation is a non-conservative force and can be considered in several ways, one of the most common is to consider that damping is proportional to velocity of displacements, known as viscous damping.

The energy dissipation of a finite element can be written as:

$$R^{(e)} = \frac{1}{2}\iiint_{V^{(e)}} \xi \dot{\hat{\mathbf{u}}}_{(e)}^{T} \dot{\hat{\mathbf{u}}}_{(e)}\, dV \tag{16}$$

Where $\xi$is the distributed damping coefficient.

Substituting Eq. (1) in Eq. (17), the damping of a finite element is given by:

$$R^{(e)} = \frac{1}{2}\iiint_{V^{(e)}} \xi \dot{\mathbf{u}}^{T}\mathbf{N}^{T}\mathbf{N}\dot{\mathbf{u}}\, dV = \frac{1}{2}\dot{\mathbf{u}}^{T}\iiint_{V^{(e)}} \xi \mathbf{N}^{T}\mathbf{N}\, dV\, \dot{\mathbf{u}} = \frac{1}{2}\dot{\mathbf{u}}^{T}\mathbf{c}\dot{\mathbf{u}} \tag{17}$$

Where the damping matrix of a finite element is:

$$\mathbf{c} = \iiint_{V^{(e)}} \xi \mathbf{N}^{T}\mathbf{N}\, dV \tag{18}$$

The global mass (**M**), stiffness (**K**) and damping (**C**) matrix, as well as the global vector loads are assembled by adding or combining the contribution of all elements that are connected to a specific node. This matrix can be written as:

$$\mathbf{M} = \sum_{e=1}^{N_e} \mathbf{m} \tag{19}$$

$$\mathbf{K} = \sum_{e=1}^{N_e} \mathbf{k} \tag{20}$$

$$\mathbf{C} = \sum_{e=1}^{N_e} \mathbf{c} \tag{21}$$

$$\mathbf{F} = \sum_{e=1}^{N_e} \mathbf{f} \tag{22}$$

Finally, the system of coupled ordinary differential dynamic equilibrium equations is given by:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F} \tag{23}$$

Before solving it, the boundary conditions and initial conditions should be applied. Equations (23) can be solved directly or by using the modal superposition method to uncouple the system of equations and some direct numerical integration method can be considered (Newmark, Central difference, Runge Kutta methods) (Bathe and Wilson [1976]).

3 Modal Analysis

A modal analysis of a system of multiple degrees of freedom (MDOF) provides the system characteristic features which are its natural vibration frequencies and associated modes of vibration. These properties are intrinsic to the system and are related to its material and geometry.

The modal analysis can be performed experimentally through tests where the system is monitored through sensors, such as accelerometers, and undergoes an external disturbance as a source of excitation.

Natural frequencies and associated vibration modes can still be determined numerically. The solution of the free vibration problem provides these characteristics of the system. There is no action by outside forces and the movement is governed only by the initial conditions.

An undamped MDOF system subjected to free vibration is governed by the following equations of motion:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0} \tag{24}$$

The system is subjected to the following set of initial conditions:

$$\mathbf{u} = \mathbf{u}(0); \quad \dot{\mathbf{u}} = \dot{\mathbf{u}}(0) \tag{25}$$

This system has a solution in the form:

$$\mathbf{u} = \boldsymbol{\phi}\sin(\omega t + \phi) \tag{26}$$

Substituting Eq. (26) into Eq. (24), we obtain:

$$(\mathbf{K} - \omega^2\mathbf{M})\boldsymbol{\phi} = \mathbf{0} \tag{27}$$

This problem has a trivial solution $\tilde{\mathbf{u}} = \mathbf{0}$ and only has non-trivial solutions if:

$$(\mathbf{K} - \omega^2\mathbf{M}) = \mathbf{0} \tag{28}$$

Eq. (28) is known as the characteristic equation and the roots of the characteristic equation determine the n natural frequencies ω_n (eigenvalues). For each ω_n there is a corresponding $\boldsymbol{\phi}$ vector (eigenvector). The eigenvectors determine the natural modes of vibration ϕ_n.

The modal matrix $\boldsymbol{\Phi}$ is constructed with n columns, where each column corresponds to a mode of vibration of the system. The fundamental mode is associated with the lowest frequency, meanwhile the other modes are called harmonics and the movement of the system is given by the superposition of the harmonics.

3.1 Modal orthogonality

The vibration modes of a MDOF system have a very important property called orthogonality. The orthogonality of the eigenvector system is with respect to the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$. This property constitutes the basis for one of the most important methods of solving dynamical systems: the Modal Superposition Method.

Rewriting Eq. (27) in the form

$$\boldsymbol{K\phi} = \omega^2 \boldsymbol{M\phi} \tag{29}$$

Considering two pairs of different solutions ($\omega_r^2, \boldsymbol{\phi}_r$ e $\omega_s^2, \boldsymbol{\phi}_s$), eq. (29) can be rewritten as:

$$\boldsymbol{K\phi}_r = \omega_r^2 \boldsymbol{M\phi}_r \tag{30}$$

$$\boldsymbol{K\phi}_s = \omega_s^2 \boldsymbol{M\phi}_s \tag{31}$$

Pre-multiplying both sides of eq. (30) by $\boldsymbol{\phi_s^T}$ then:

$$\boldsymbol{\phi_s^T K\phi}_r = \omega_r^2 \boldsymbol{\phi_s^T M\phi}_r \tag{32}$$

Similarly for equation (31) multiplying both sides by $\boldsymbol{\phi_r^T}$ comes to

$$\boldsymbol{\phi_r^T K\phi}_s = \omega_s^2 \boldsymbol{\phi_r^T M\phi}_s \tag{33}$$

As the matrices $\boldsymbol{M}$ and $\boldsymbol{K}$ are symmetric, then $\boldsymbol{K} = \boldsymbol{K}^{\mathrm{T}}$ and $\boldsymbol{M} = \boldsymbol{M}^{\mathrm{T}}$. Therefore equation (33) can be transposed and rewritten as

$$\boldsymbol{\phi_s^T K\phi}_r = \omega_s^2 \boldsymbol{\phi_s^T M\phi}_r \tag{34}$$

Equation (32) can be subtracted from eq. (33) provides

$$(\omega_r^2 - \omega_s^2)\boldsymbol{\phi_s^T M\phi}_r = 0 \tag{35}$$

For modes with different frequencies $\omega_r \neq \omega_s$it is necessary that

$$\boldsymbol{\phi_s^T M\phi_r} = 0 \tag{36}$$

The r-th and s-th modes are said to be orthogonal to the mass matrix. Equation (36) can be substituted into Equation (32) to show that the r-th mode and the s-th mode are also orthogonal with respect to the stiffness matrix:

$$\boldsymbol{\phi_s^T K\phi_r} = 0 \tag{37}$$

These mode orthogonality conditions are the essence of the Modal Superposition Method.

3.2 Mode normalization

The vibration amplitudes in a mode are just relative values that can be scaled or normalized to some extent, as a matter of choice. A convenient form of normalization for a general system is:

$$\boldsymbol{\phi}'_{ij} = \frac{\boldsymbol{\phi_{ij}}}{\sqrt{\boldsymbol{\phi}_i^T \boldsymbol{M\phi}_j}} \tag{38}$$

i= i-th element of the eigenvector

j= jth vibration mode

For a system with a diagonal mass matrix

$$\boldsymbol{\phi}'_{ij} = \frac{\boldsymbol{\phi}_{ij}}{\sqrt{\sum_{k=1}^{n} m_k \phi_{kj}^2}} \tag{39}$$

$$\boldsymbol{\phi}'_{ij} = \frac{\boldsymbol{\phi}_{ij}}{\sqrt{\boldsymbol{\phi}_j^{\top} \boldsymbol{M} \boldsymbol{\phi}_j}} \tag{40}$$

For normalized eigenvectors the orthogonality condition is given by:

$$\begin{aligned} \boldsymbol{\phi}'^{T}_i \boldsymbol{M} \boldsymbol{\phi}_j &= 0 \text{ with } i \neq j \\ &= 1 \text{ with } i = j \end{aligned} \tag{41}$$

Another orthogonality condition is obtained by writing Equation (29) for the normalized mode j as:

$$\boldsymbol{K} \boldsymbol{\phi}'_j = \omega_j^2 \boldsymbol{M} \boldsymbol{\phi}'_j \tag{42}$$

Pre-multiplying eq. (42) by $\boldsymbol{\phi}_i^T$ we obtain the following orthogonality condition between the eigenvectors

$$\begin{aligned} \boldsymbol{\phi}'^{T}_i \boldsymbol{K} \boldsymbol{\phi}_j &= 0 \quad \text{with } i \neq j \\ &= \omega_j^2 \text{ with } i = j \end{aligned} \tag{43}$$

4 Numerical Solution for Natural Frequencies and Vibration Modes

A critical step in the dynamic analysis of a MDOF system is the solution or determination of the corresponding natural frequencies and associated vibration modes. Especially in the case of an analysis by Modal Superposition.

For minor problems the solution to this problem lies in the determination of a characteristic polynomial. For high-order MDOF systems, extracting these roots manually becomes prohibitive.

Over the years, several numerical techniques have been developed to solve this type of problem. They are essentially iterative techniques and can mean a considerable computational effort. The present mathematical problem is given by:

$$\boldsymbol{K\Phi = \lambda M \phi} \tag{44}$$

where the eigenvalues are $\lambda_r = \omega^2$ and the eigenvectors are $\boldsymbol{\phi}_r$.

The response of a MDOF system with a large number of degrees of freedom is generally restricted to a relatively small subset of the system's lowest vibration modes. Thus, it is only necessary to solve for a number p of eigenvalue / eigenvector sets.

Most solution techniques for this type of problem can be classified into 3 basic categories:

- Vector iteration methods
- Transformation methods
- Polynomial iteration methods.

The most common eigenproblems that are encountered in general scientific analysis are standard eigenproblems, and most other eigenproblems can be reduced to a standard form.

For this reason, the solution of standard eigenproblems has attracted much attention in numerical analysis, and many solution algorithms are available.

When the mode superposition method is performed, the computational effort relays on obtaining natural frequencies and associated vibration modes. Effective numerical algorithms for this calculation have been widely studied since the exact solution of this type of problem is prohibitive when the system order is large.

In the literature it can be found several eigenvalue/eigenvector solution algorithms. They are generally formulated for ordinary matrices. When working on a finite element method perspective the matrices involved have peculiarities, such as being banded, positive definite among others. It is recommendable that the solution techniques benefit from these properties searching a more economic solution.

The approximate solution techniques have primarily been developed to calculate the lowest eigenvalues and corresponding eigenvectors in the Eq. (44), when the order of the system is large. Most programs use exact solution techniques in the analysis of small-order systems. However, the problem of calculating the few lowest eigenpairs of relatively large-order systems is very important and is encountered in all branches of structural engineering. In the following sections we present three major techniques, all of which, in fact, can be considered to be a Ritz analysis.

Consider that we have mass and stiffness matrices, $\boldsymbol{K}$ and $\boldsymbol{M}$, definite positives. The consistent or lumped mass matrix has all the elements outside of the diagonal different from zero. The eigenvalue calculated by Ritz analysis is an upper limit of the system exact eigenvalue. The associated error is not taken into account. This error will depend on the Ritz vector base considered, since the eigenvector $\boldsymbol{\phi}$ is a linear combination of this vector base.

The Ritz procedure is a very general one, various analysis methods known by different names can be classified as Ritz analysis. One of the most important aspects in the analysis is the base vector choice.

All the methods are iterative by nature since they try to solve the eigenvalue problem that is equivalent to calculate the roots of the characteristic polynomial $P(\lambda)$ that has the same order of the mass and stiffness matrices. There is no exact solution to polynomials with order higher than four, so it is necessary to perform iterative techniques.

What can be done before the iterative method is to transform $\boldsymbol{K}$ and $\boldsymbol{M}$ in a way to allow a more economic computational effort, one example is the static condensation technique.

There is no such as an unique algorithm that gives an efficient solution. The method accuracy depends considerably on two factors: reliability and computational effort.

4.1 Vector iteration methods

The philosophy of this family method is to satisfy the equation (44) by operating directly on it. It is assumed a vector $\mathbf{x}_1$ for $\boldsymbol{\phi}$, and assume the associated eigenvalue $\boldsymbol{\lambda}$ equal to $\mathbf{1}$. Equation (44) can be rewritten as

$$\mathbf{R}_1 = \mathbf{1}\boldsymbol{M}\mathbf{x}_1 \tag{45}$$

$\mathbf{R}_1$ is not yet determined since not necessarily an eigenvector. There is an equilibrium equation given by

$$\mathbf{K}\mathbf{x}_2 = \mathbf{R}_1, \mathbf{x}_2 \neq \mathbf{x}_1 \tag{46}$$

$\mathbf{x}_2$ is the displacement solution to the applied forces $\mathbf{R}_1$. It means to be a better approximation to the eigenvector than $\mathbf{x}_2$, repeating the procedure several times it can be obtained a superior approximation to $\boldsymbol{\phi}$.

This methodology is the way of work basically of inverse iteration. This procedure has a good performance on obtaining an eigenvector and also the eigenvalue associated.

Assume that $\mathbf{K}$ is positive definite and $\mathbf{M}$ is a diagonal matrix without zero diagonal elements. Assume an initial iteration vector $\mathbf{x}_1$ and then evaluate in each iteration step k = 1,2,...

$$\mathbf{K}\bar{\mathbf{x}}_{k+1} = \mathbf{M}\mathbf{x}_k \tag{47}$$

and

$$\mathbf{x}_{k+1} = \frac{\bar{\mathbf{x}}_{k+1}}{\left(\bar{\mathbf{x}}_{k+1}^T \mathbf{M} \bar{\mathbf{x}}_{k+1}\right)^{1/2}} \tag{48}$$

It has to be assured that $\mathbf{x}_1^T \mathbf{M} \boldsymbol{\phi}_1 \neq 0$, in other words $\mathbf{x}_1$ is not orthonormal to $\boldsymbol{\phi}_1$, so we can assume that

$$\mathbf{x}_{k+1} \longrightarrow \phi_1 \text{ as } k \longrightarrow \infty \tag{49}$$

Equation (47) has to be solved evaluating a vector $\bar{\mathbf{x}}_{k+1}$ getting closer to an eigenvector compared to the preceding iteration $\mathbf{x}_k$. According to (48) it is needed that the orthonormality relation above is complied

$$\mathbf{x}_{k+1}^T \mathbf{M} \mathbf{x}_{k+1} = 1 \tag{50}$$

The equations (47) and (48) depict the basic inverse iteration algorithm. for computational implementation we can consolidate in the form

$$\mathbf{K}\bar{\mathbf{x}}_{k+1} = \mathbf{y}_k \tag{51}$$

$$\bar{\mathbf{y}}_{k+1} = \mathbf{M}\bar{\mathbf{x}}_{k+1} \tag{52}$$

$$\rho\left(\bar{\mathbf{x}}_{k+1}\right) = \frac{\bar{\mathbf{x}}_{k+1}^T \mathbf{y}_k}{\bar{\mathbf{x}}_{k+1}^T \bar{\mathbf{y}}_{k+1}} \tag{53}$$

$$\mathbf{y}_{k+1} = \frac{\bar{\mathbf{y}}_{k+1}}{\left(\bar{\mathbf{x}}_{k+1}^T \bar{\mathbf{y}}_{k+1}\right)^{1/2}} \tag{54}$$

Consider that $\mathbf{y}_1^T \boldsymbol{\phi}_1 \neq 0$, is guaranteed,

$$\mathbf{y}_{k+1} \longrightarrow \mathbf{M}\boldsymbol{\phi}_1 \tag{55}$$

$$\rho\left(\bar{\mathbf{x}}_{k+1}\right) \longrightarrow \lambda_1 \text{ as } k \longrightarrow \infty \tag{56}$$

The Raylegh quotient $\rho\left(\bar{\mathbf{x}}_{k+1}\right)$ gives an approximation to the eigenvalue λ_1 that is considered to set up the convergence criterion

$$\frac{\left|\lambda_1^{(k+1)} - \lambda_1^{(k)}\right|}{\lambda_1^{(k+1)}} \leq \text{tol} \tag{57}$$

For the last iteration we have

$$\lambda_1 \doteq \rho\left(\bar{\mathbf{x}}_{l+1}\right) \tag{58}$$

and

$$\phi_1 \doteq \frac{\bar{\mathbf{x}}_{l+1}}{\left(\bar{\mathbf{x}}_{l+1}^T \bar{\mathbf{y}}_{l+1}\right)^{1/2}} \tag{59}$$

Transformation methods

This set of techniques called Transformation Methods include procedures that take into account basic properties of the eigenvectors in matrix **Φ,**

$$\mathbf{\Phi}^T \mathbf{K}\, \mathbf{\Phi} = \mathbf{\Lambda} \tag{60}$$

$$\mathbf{\Phi}^T \mathbf{M} \mathbf{\Phi} = \mathbf{I} \tag{61}$$

The matrix **Φ** that diagonalize **K** and **M** is unique, it can be found iteratively.

We have to reduce **K** and **M** to their diagonal way pre and post multiplying them by the matrices $\mathbf{P}_k^T$ and $\mathbf{P}_k$, respectively where k = 1,2,… It starts with $\mathbf{K}_1 = \mathbf{K}$ and $\mathbf{M}_1 = \mathbf{M}$, and it follows:

$$\begin{aligned} \mathbf{K}_2 &= \mathbf{P}_1^T \mathbf{K}_1 \mathbf{P}_1 \\ \mathbf{K}_3 &= \mathbf{P}_2^T \mathbf{K}_2 \mathbf{P}_2 \\ &\cdot \\ &\cdot \\ \mathbf{K}_{k+1} &= \mathbf{P}_k^T \mathbf{K}_k \mathbf{P}_k \end{aligned} \tag{62}$$

In the same way

$$\begin{aligned} \mathbf{M}_2 &= \mathbf{P}_1^T \mathbf{M}_1 \mathbf{P}_1 \\ \mathbf{M}_3 &= \mathbf{P}_2^T \mathbf{M}_2 \mathbf{P}_2 \\ &\cdot \\ &\cdot \\ \mathbf{M}_{k+1} &= \mathbf{P}_k^T \mathbf{M}_k \mathbf{P}_k \end{aligned} \tag{63}$$

To converge the process, it is necessary that

$$\begin{gathered}\mathbf{K}_{k+1} \longrightarrow \mathbf{\Lambda} \\ \mathbf{M}_{k+1} \longrightarrow \mathbf{I} \\ k \longrightarrow \infty\end{gathered} \tag{64}$$

The last iteration is given by:

$$\mathbf{\Phi} = \mathbf{P}_1\mathbf{P}_2 \ldots \mathbf{P}_l \tag{65}$$

It is not really necessary that $\mathbf{M}_{k+1} \longrightarrow \mathbf{I}$ and $\mathbf{K}_{k+1} \longrightarrow \mathbf{\Lambda}$ but yes that they converge to a diagonal form. Based on the methodology described above several iteractive methods have been proposed such as Jacobi and the Householder-QR methods, very effective in finite element analysis.

Polynomial iteration techniques

The eigenvalue and eigenvector problem lies in the solution of calculating the zeros of a polynomial of degree *n,* known as a characteristic polynomial, given by

$$P(\lambda) = \det(\mathbf{K} - \lambda\mathbf{M}) \tag{66}$$

The characteristic polynomial roots represent the eigenvalues of the eigenproblem

$$\mathbf{K}\boldsymbol{\phi} = \lambda\mathbf{M}\boldsymbol{\phi} \tag{67}$$

Solving this problem relay in two different strategies: explicit and implicit evaluation procedures, which can use the same basic iteration schemes. It directly uses the $\boldsymbol{K}$ and $\boldsymbol{M}$ matrices from finite element modeling without transforming the problem to a different form. Only the eigenvalues are obtained and the eigenvectors can be obtained by inverse iteration with substitution.

Explicit polynomial iteration

The characteristic polynomial is written as

$$P(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n \tag{68}$$

and the polynomial coefficients a_0, a_1,…, a_n are evaluated. Then the polynomial roots are calculated. Small errors in the coefficients cause considerable errors in the roots of the polynomial. For this reason this method fell into disuse for the eigenproblem solution.

Implicit polynomial iteration

In this method the value of the characteristic polynomial is evaluated directly without the need to calculate the coefficients. The value of $P(\lambda)$ can be obtained effectively by decomposing $\mathbf{K} - \lambda\mathbf{M}$ into a lower triangular matrix $\boldsymbol{L}$ and an upper triangular matrix $\boldsymbol{S}$:

$$\mathbf{K} - \lambda\mathbf{M} = \mathbf{LS} \tag{69}$$

$$\det(\mathbf{K} - \lambda\mathbf{M}) = \prod_{i=1}^{n} s_{ii} \tag{70}$$

5 Modal Superposition Method

If the main vibration modes of a MDOF system are used as a generalized coordinate system to define the motion response, the *n* motion equations become decoupled. In these independent coordinates, each uncoupled equation can be solved independently, as if each equation belonged to an independent single degree of freedom system.

Therefore, the individual responses of the *n* MDOF system decoupled equations (one for each vibration mode) for any form of excitation can be determined by applying analysis techniques to onde degree of freedom (1DOF) system. The modal superposition MDOF system response is then defined by the sum of the individual mode responses.

The Modal Superposition Method is a very popular vibration analysis technique, but it has some important limitations. The method is valid only for linear and proportional damping systems. This is most useful when the response can be accurately evaluated considering only a relatively small subset of system modes.

It should be noted, however, that for most dynamic loadings on structural and mechanical systems, the contributions of the various modes to the dynamic response are generally more considerable at lower frequencies and tend to decay at higher frequencies.

Consequently, it is not necessary to include the higher modes in the superposition process.

The motion equation for a structural system subjected to dynamic excitations is given by

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{P}(\mathrm{t}) \tag{71}$$

Where **M**, **C** and **K** are the mass, damping and stiffness matrices of the structure and P(t) is the dynamic loading vector. $\ddot{\boldsymbol{x}}$, $\dot{\boldsymbol{x}}$ e $\boldsymbol{x}$ are the vectors of acceleration, velocities and displacements, respectively.

The natural frequencies ω_i and the vibration modes $\boldsymbol{\phi}_i$ corresponding to each i-th degree of freedom of the structure (i = 1,2,...,N) can be obtained by solving the eigenvalue problem

$$\mathbf{K}\boldsymbol{\phi}_i = \omega_i^2 \mathbf{M}\boldsymbol{\phi}_i \tag{72}$$

The matrix whose columns are the modal forms $\boldsymbol{\phi}_i$ is called $\boldsymbol{\Phi}$, which presents orthogonality properties in relation to the structure's mass and stiffness matrices.

The modes can then be normalized such that the generalized mass and stiffness matrices are:

$$\overline{\mathbf{M}} = \Phi^{\mathrm{T}}\mathbf{M}\Phi = \mathbf{I} \tag{73}$$

$$\overline{\mathbf{K}} = \Phi^{\mathrm{T}}\mathbf{K}\Phi \tag{74}$$

$\overline{\boldsymbol{K}}$ is a diagonal matrix whose diagonal elements are the squared natural frequencies.

Through the modal matrix the physical coordinates **x** can be transformed into generalized coordinates **y** through the equation

$$\mathbf{x} = \boldsymbol{\Phi}\mathbf{y} \tag{75}$$

The equation of motion in modal coordinates is expressed by

$$\overline{\mathbf{M}}\ddot{\mathbf{y}} + \overline{\mathbf{C}}\dot{\mathbf{y}} + \overline{\mathbf{K}}\mathbf{y} = \overline{\mathbf{P}} \tag{76}$$

In Equation (76) the generalized damping matrix and the generalized loading vector are given, respectively by

$$\bar{\mathbf{C}} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{C}\mathbf{\Phi} \tag{77}$$

$$\bar{\mathbf{P}} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{P} \tag{78}$$

The generalized mass and stiffness matrices are diagonal matrices due to the orthogonality of the vibration modes. For the generalized damping matrix this is not necessarily true, only for the case of the proportional damping matrix

$$\mathbf{C} = a_0\mathbf{M} + a_1\mathbf{K} \tag{79}$$

In the case of proportional damping, the modal matrix $\overline{\boldsymbol{C}}$ is also diagonal and the modal equations of motion become decoupled

$$\boldsymbol{\phi}_i^T\mathbf{C}\boldsymbol{\phi}_i = \begin{cases} 2\omega_i\xi_i\bar{m}_i & i = j \\ 0 & i \neq j \end{cases} \tag{80}$$

The equation of motion of the n vibration mode is given by

$$\bar{m}_n\ddot{y}_n + 2\xi_n\omega_n\bar{m}_n\dot{y}_n + \bar{k}_n y_n = \bar{p}_n(t) \tag{81}$$

or

$$\ddot{y}_n + 2\xi_n\omega_n\dot{y}_n + \omega_n^2 y_n = \frac{\bar{p}_n}{\bar{m}_n}(t) \tag{82}$$

with

$$\begin{aligned} \bar{m}_n &= \boldsymbol{\phi}_n^{\mathrm{T}}\mathbf{M}\boldsymbol{\phi}_n \\ \bar{k}_n &= \boldsymbol{\phi}_n^{\mathrm{T}}\mathbf{K}\boldsymbol{\phi}_n \\ \bar{p}_n(t) &= \boldsymbol{\phi}_{\mathrm{n}}^{\mathrm{T}}\mathbf{p}(t) \\ \bar{c}_n = \boldsymbol{\phi}_n^T\mathbf{C}\boldsymbol{\phi}_n &= 2\xi_n\omega_n\bar{m}_n \end{aligned} \tag{83}$$

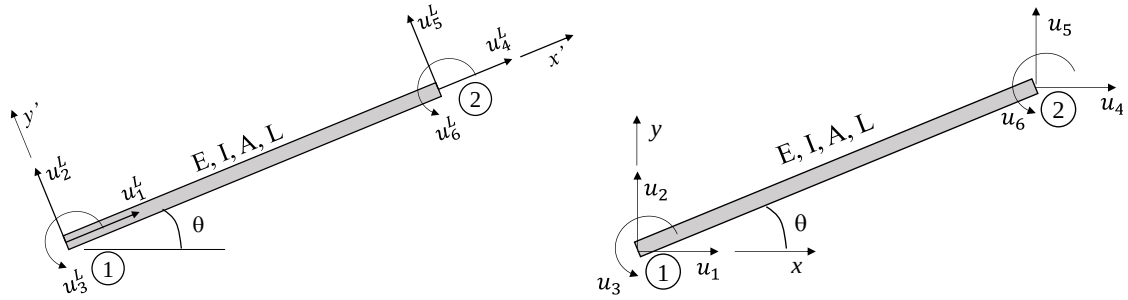
The superposition of the effects corresponding to each modal equation provides the structure response

$$\boldsymbol{x}(t) = \boldsymbol{\phi}_1 y_1(t) + \boldsymbol{\phi}_2 y_2(t) + \cdots + \boldsymbol{\phi}_n y_n(t) \tag{84}$$

It is not necessary to include all higher modes as for most loads the displacement contributions are greater for lower modes and reduced for higher modes.

6 Sensitivity Analysis in Plane Frames

Consider a beam element of length *L*, cross section *A*, inertia *I*, Young's modulus *E*, distributed mass *m* and rotated an angle θ in relation to global coordinates x-y, as seen in Fig. 2. Figure 2(a) shows the six degrees of freedom ($u_1^L, u_2^L, u_3^L, u_4^L, u_5^L$ and u_6^L) in local coordinates and Figure 2(b) shows the six degrees of freedom (u_1, u_2, u_3, u_4, u_5 and u_6) in global coordinates.



(a) (b)

Figure 2 – Beam element. (a) Local coordinates and b) Global coordinates

In order to simulate loss of stiffness at beam joints, it is possible to consider rotational springs of stiffness K_1 and K_2 located at both ends of the beam, which represent semi-rigid joints, as seen in Figure 3 (McGuire, Gallagher and Ziemian, 2000 and Paz and Kim, 2019).

Figure 3 – Beam element with rotational springs at both ends.

Each rotational spring has the following stiffness:

$$K_1 = \alpha_1 \frac{EI}{L} \tag{85}$$

$$K_2 = \alpha_2 \frac{EI}{L} \tag{86}$$

By applying shape functions and compatibility equations, the local beam stiffness matrix can be written as (Paz and Kim, 2019 and McGuire, Gallagher and Ziemian, 2000):

$$\mathbf{k} = \begin{bmatrix} a & 0 & 0 & a & 0 & 0 \\ 0 & b & c & 0 & -b & d \\ 0 & c & e & 0 & -c & f \\ a & 0 & 0 & a & 0 & 0 \\ 0 & -b & -c & 0 & b & -d \\ 0 & d & f & 0 & -d & g \end{bmatrix} \tag{87}$$

Where:

$$a = \frac{EA}{L} \qquad\qquad e = 4\frac{EI}{L}\alpha\left(1 + \frac{3}{\alpha_2}\right) \tag{88}$$

$$b = 12\frac{EI}{L^3}\alpha\left(1+\frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2}\right) \qquad f = 2\frac{EI}{L}\alpha$$

$$c = 6\frac{EI}{L^2}\alpha\left(1+\frac{2}{\alpha_2}\right) \qquad g = 4\frac{EI}{L}\alpha\left(1+\frac{3}{\alpha_1}\right)$$

$$d = 6\frac{EI}{L^2}\alpha\left(1+\frac{2}{\alpha_1}\right) \qquad \alpha = \frac{\alpha_1\alpha_2}{\alpha_1\alpha_2+4\alpha_1+4\alpha_2+12}$$

The local beam consistent mass matrix, can be written as:

$$\mathbf{m} = \rho\frac{AL}{420}\begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22L & 0 & 54 & -13L \\ 0 & 22L & 4L^2 & 0 & 13L & -3L^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13L & 0 & 156 & -22L \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 \end{bmatrix} \tag{89}$$

The global stiffness and mass matrix can be obtained by applying the rotation matrix as:

$$\mathbf{K} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{k}\mathbf{\Phi} \tag{90}$$

$$\mathbf{M} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{m}\mathbf{\Phi} \tag{91}$$

Where the rotation matrix can be written as:

$$\mathbf{\Phi} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{92}$$

To study the effect of loss of stiffness in some joint on the natural frequencies of a system, consider a plane frame as seen in Figure 4.

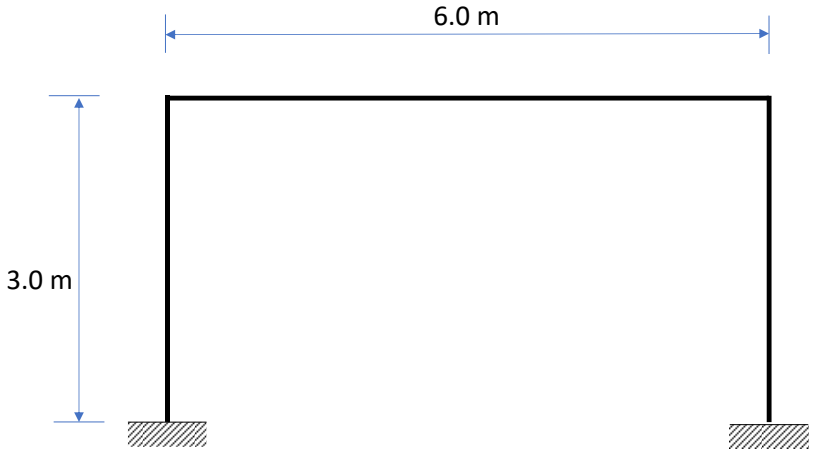


Figure 4 – Plane frame.

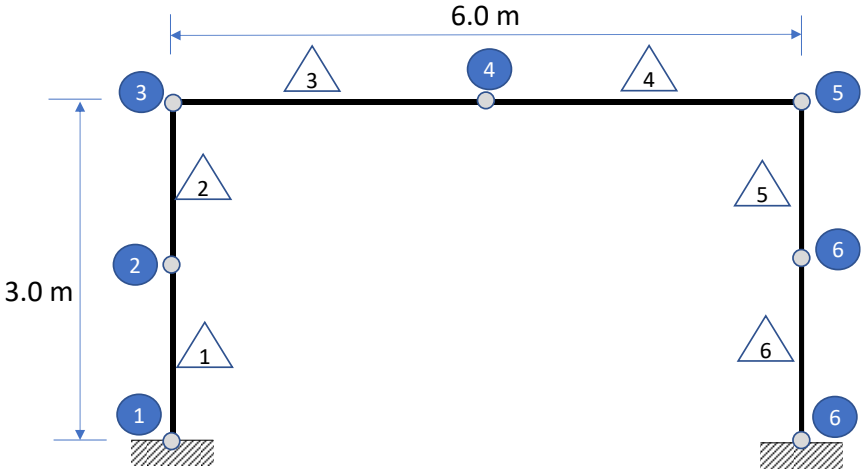
The frame has the following geometric and physical properties:

Young´s modulus (E) = 250e5 N/m^2, mass density (ρ) = 3000 kg/m^3.

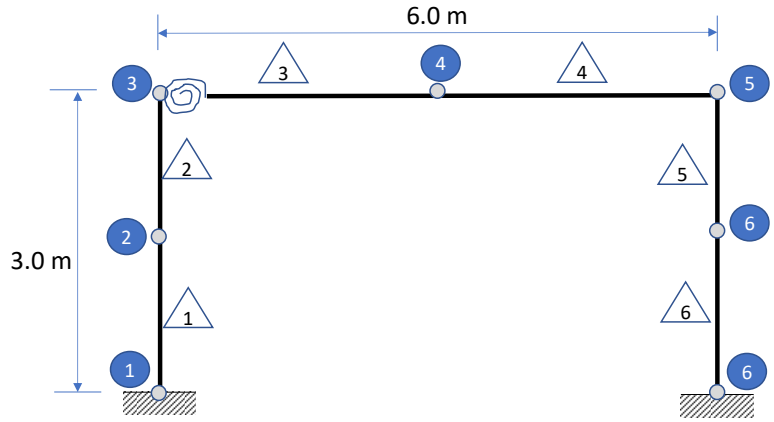
Beam: Cross section (A) = 0.08 m^2, Moment of inertia (I) = 1.0667e-3 m^4.

Columns: Cross section (A) = 0.048 m^2, Moment of inertia (I) = 6.4e-4 m^4.

The frame was modeled with six beam elements and six nodes as seen in Fig. 5(a) and, at the left joint of element three, a rotational spring was considered to model the loss of stiffness due to reduction of stiffness on the cross section as displayed in Fig. 5(b).



(a)



(b)

Figure 5 – Discretized plane frame

To study the influence of loss of stiffness on the natural frequencies and vibration modes of the plane frame, the stiffness of rotational spring was varied. Figure 6 displays the first three natural frequencies and associated vibration modes s the value of α_1 is reduced. As can be seen, there reduction of α_1 generates a reduction the natural frequencies as well as changes on the shape of vibration modes.

Figure 7 displays the variation of the normalized first three natural frequencies for variation of α_1, and as can be observed, the variation of α_1 parameter, does not affect linearly the natural frequencies but, it affects in a non-linear relation. For high values of α_1, the frame is considered with rigid joints and as this parameter is reduced, there is also a reduction of the natural frequencies. All natural frequencies are affected by reduction of α_1 parameter and, it is possible to see, that the first natural frequency is more than second and third natural frequencies.

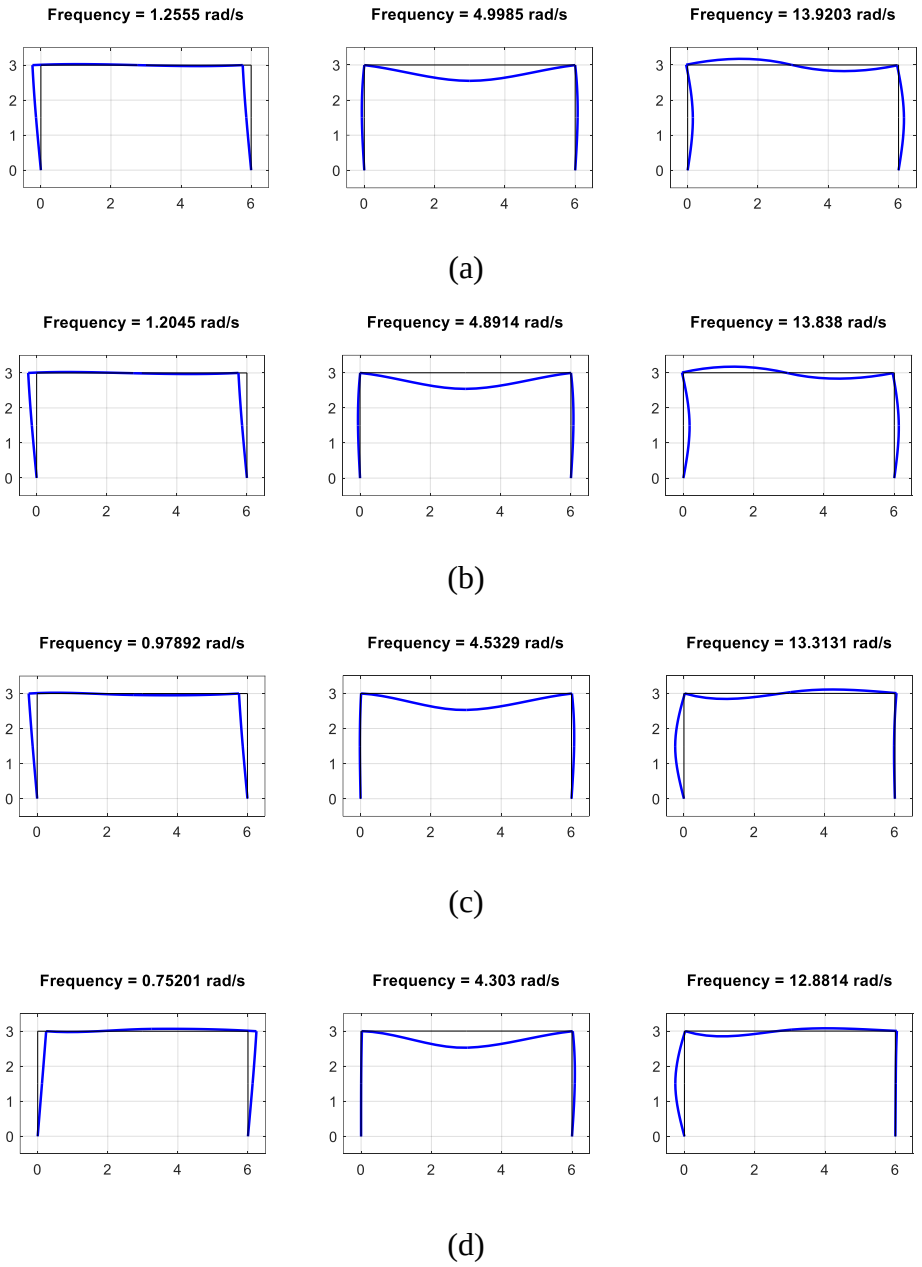


Figure 6: First natural frequencies and associated vibration modes for variation of α_1. a) α_1 = 1000.0, b) α_1 = 10.0, c) α_1 = 1.0 and d) α_1 = 0.01.

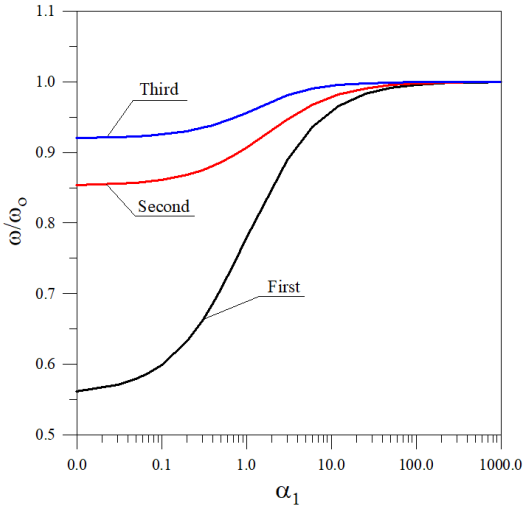


Figure 7: Variation of first natural frequencies for variation of α_1.

7 Concluding Remarks

In this chapter the application of the Finite Element Method to structural dynamics problems was addressed and topics such as modal analysis, modal superposition method were presented. Also, numerical methods for solving eigenvalue and eigenvector problems were briefly discussed. At the end, results of the computational simulation of a structure are presented, considering the effects generated on the natural frequencies of vibration and vibration modes when there is a loss of stiffness in the connections: a sensitivity analysis.

This subject is a wide field for future studies and research, among them we can mention the application of inverse problems and uncertainty modeling techniques in the variables of the problem.

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