Chapter 14

Time Series Models and Expansion Optimal Linear Estimation for Time-Dependent Reliability Analysis

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Time Series Models and Expansion Optimal Linear Estimation for Time-Dependent Reliability Analysis

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Abstract

This chapter presents conceptual and computational comparisons between EOLE (Expansion Optimal Linear Estimation) and AR (Auto-Regressive) models to represent stochastic processes in the context of time-dependent reliability analysis. Even though expansion techniques, such as EOLE, are appropriate for problems where the properties of the stochastic process are explicitly known, such information is rarely available in practical situations. On the other hand, time series models, such as AR, are widely employed to represent stochastic processes from real time monitoring or available historical data. For this reason, here we compare EOLE and AR in the context of time-dependent reliability analysis. We first demonstrate how AR models can be calibrated to represent a given stochastic process with known properties. It is then demonstrated that similar results for time-dependent reliability can be obtained using the two approaches. This is an important contribution from both conceptual and practical reasons, since it demonstrates that existing AR models (and likely other types of time series models) can be directly employed for time-dependent reliability analysis, without the need to first obtain an equivalent EOLE model.

Keywords: Time Dependent Reliability Analysis; Expansion Optimal Linear Estimation; Time Series Models;

1 Introduction

Several problems from sciences and engineering cannot be appropriately studied in the deterministic context, due to uncertainties concerning its parameters. In these cases, some kind of probabilistic (i.e. stochastic) approach is necessary. In this kind of approach, some problem parameters are modeled as random variables and/or stochastic process, in order to take into account randomness and uncertainty.

A problem of great interest in engineering is that of Reliability Analysis. In this context, one wishes to evaluate the probability of occurrence of undesired events, generally classified as failures. Reliability Analysis is widely employed for design considering uncertainties in several fields of engineering, such as: aerospace, nuclear, structural, mechanical, among others.

Several engineering systems of interest are also subject to uncertain time-dependent effects, such as ageing, deterioration, corrosion and time-evolving demands. In these situations, it is necessary to model the time-dependent uncertainties as Stochastic Processes. This approach leads to what is generally known as Time-dependent Reliability Analysis, a topic of great interest in the scientific community.

Most works concerning time-dependent Reliability Analysis presented in the past employed the same technique to represent the involved stochastic processes: the Expansion Optimal Linear Estimation (EOLE). EOLE was originally presented by Li and Kiureghian [1993] and is very popular because of its generality and computational efficiency. For this reason, most available methods for time-dependent Reliability Analysis are based on the concepts of EOLE. However, it should be noted that EOLE is not, by far, the most popular technique employed to model stochastic process in practice. In fact, the most common models employed in practice are those generally known as Time Series Models, such as: Auto-Regressive (AR), Moving Average (MA), Auto-Regressive with Moving Average (ARMA) and Auto-Regressive Integrated Moving Average (ARIMA) [Cryer and Chan, 2008, Box and Jenkins, 1978]. The ARMA model, in particular, is widely employed in several fields of sciences and engineering to model real time-dependent data. In the engineering community things are not much different. Time Series Models such as AR and ARMA are much more popular in practical applications concerning modeling of time-dependent data.

The popularity of Time Series Models[1] is likely a consequence of the simple mathematical structure of such models. As will be discussed later on this work, the AR model has a much simpler mathematical structure than EOLE. For this reason, Time Series Models are generally in the reach of a larger number of scientists and engineers. However, it must be emphasized that another apparent reason for the popularity of Time Series Models is that this subject has been under careful investigation by researchers for almost a century now. Thus, there are sound mathematical foundations and efficient computational techniques concerning these methods.

Even though Time Series Models were already employed in the context of time-dependent Reliability Analysis, specific comparison to EOLE was only carried out in the works by Kroetz et al. [2020] and de Medeiros [2022]. For this reason, in this chapter we compare the application of Time Series Models (AR in particular) and EOLE in the context of time-dependent Reliability Analysis. Here we wish do answer two main questions: i) how can one apply the AR model for problems that have been traditionally modeled using EOLE; ii) are the results obtained with AR models and EOLE similar? In this work we only investigate stationary Stochastic Process, that can be modeled using the AR model. Also, we do not try to discuss the possible advantages of employing Time Seris Models over EOLE, since we understand that this depends much more on the application being investigated.

[1] Here we employ the term "Time Series Models" to collectively represent the MA, AR, ARMA, ARIMA models and its modifications.

The rest of this work is organized as follows. In the next section we present an overview of the literature concerning the subject of this work. We then present the main concepts concerning Reliability Analysis and time-dependent Reliability Analysis. Application of Monte Carlo Simulation for evaluation of the probability of failure is detailed in Section 5. The EOLE and the AR techniques for modeling a stochastic process are presented in Sections 6 and 7, respectively. We then present two numerical examples to demonstrate the results obtained. The concluding remarks of this work are finally summarized in Section 9.

2 Literature Overview

In the context of time-dependent Reliability Analysis, stochastic processes are frequently modeled using expansion techniques such as Karhunen-Loève (KL) and Expansion Optimal linear estimation method (EOLE). The KL expansion (see Loève [1977]) is widely employed in several fields of science and engineering. However, the EOLE, originally presented by Li and Kiureghian [1993], presents some numerical advantages in comparison to KL, mainly for the representation of stochastic process with arbitrary distribution. For this reason, EOLE is more popular in the context of time-dependent Reliability Analysis. Zhang and Ellingwood [1994] also compared expansion techniques in the context of Reliability Analysis.

An important method for time-dependent Reliability Analysis is the PHI2, originally proposed by Andrieud-Renaud et al. [2004] and latter improved by Sudret [2008]. The PHI2 method basically estimates an upper bound for the probability of failure by integration of the failure rate. The failure rate, on the other hand, is evaluated using concepts from FORM (First Order Reliability Method) [Madsen et al., 1986, Ditlevsen and Madsen, 1996a]. Some additional methods on the subject were later proposed by Li and Mourelatos [2009], Singh et al. [2010], Hu and Du [2013a], Hu and Du [2013b], Hu and Du [2014], Jiang et al. [2014], Jiang et al. [2017], Jiang et al. [2018], Hawchar et al. [2015], Hawchar et al. [2017], Moarefzadeh and Sudret [2018], Nie and Ellingwood [2005], Gong and Frangopol [2019], Zhou et al. [2017], Yu et al. [2020], Wu et al. [2020], Hu et al. [2021], Zhang et al. [2021]. However, it must be emphasized that all these methods are based on a discretization of the stochastic processes using expansion techniques such as EOLE or KL.

The works by Singh et al. [2011a], Singh et al. [2011b], Walls and Bendell [1987], Mignolet and Spanos [1989], Ho and Xie [1998], Billinton and Wangdee [2007] Sohn et al. [2000], Fugate et al. [2001], Sohn and Farrar [2001], Sohn and Farrar [2001], Owen et al. [2001] Omenzetter and Brownjohn [2006], Carden and Brownjohn [2008], Zheng and Mita [2008], Drignei [2011], Bao et al. [2013], Yao and Pakzad [2013], Goyal and Pabla [2016], Hu and Mahadevan [2017], Datteo et al. [2018], Zhang et al. [2018], Wang et al. [2021], Samaras et al. [1985] and Hu and Mahadevan [2017] employed Time Series models (i.e. AR, ARMA, among others) in the context of structural engineering and Time-dependent Reliability Analysis. These works demonstrate the popularity of Time Series models in practical engineering applications. However, no extensive comparison between Time Series models and EOLE in the context of Reliability Analysis was carried until recently. The work by Kroetz et al. [2020] was the first to extensively compare EOLE and ARMA in the context of time-dependent Reliability analysis. However, in that work the employed ARMA models were calibrated using a sample obtained with EOLE and standard techniques for Time Series calibration, described by Cryer and Chan [2008], Box and Jenkins [1978]. Although the comparisons were enough to demonstrated that ARMA was able to obtain similar results to EOLE, the calibration approach employed included sampling error. The work by Kroetz et al. [2020] was latter extended in the thesis by de Medeiros [2022], were a more accurate approach was conceived to calibrate ARMA models directly from the properties of the stochastic processes. The approach employed by de Medeiros [2022] to calibrate the ARMA is based on an optimization scheme that searches for the time series coefficients that minimize the distance between the resulting ARMA autocovariance and the autocovariance of the stochastic process. The approach also includes constraints on stationarity and invertibility of the resulting ARMA time series.

This chapter is based on the works by Kroetz et al. [2020] and de Medeiros [2022]. However, here we focus on AR models, that are generally simpler to employ and can be calibrated by solving a system of linear equations.

3 Reliability Analysis

Consider a limit state function g that depends on a vector of random variables $\mathbf{X}$, such that $g < 0$ indicates failure of the system under analysis. We then define the probability of failure as [Madsen et al., 1986, Ditlevsen and Madsen, 1996b, Melchers and Beck, 2018, Beck, 2019]

$$P_f = P\left[g\left(\mathbf{X}\right) < 0\right], \tag{1}$$

where $P[\cdot]$ represents the probability of ocurrence of a given event.

The most common type of limit state function that appears in Structural Safety is of the kind

$$g = R - S, \tag{2}$$

where R and S represent the resistance and the load effects, respectively. In this case, failure is assumed to occur when $S > R$. Several problems in engineering can be stated in this form, considering that R represents the capacity of the system under analysis and S represents the demand that the system must supply.

Assuming the random vector $\mathbf{X}$ has density given by $f_X(\mathbf{x})$ with support $\Omega \subseteq \mathbb{R}^n$, the probability of failure can also be written as

$$P_f = \int_\Omega I\left(g\left(\mathbf{x}\right)\right) f_X\left(\mathbf{x}\right) d\mathbf{x}, \tag{3}$$

where

$$I(\zeta) = \begin{cases} 1, & \zeta < 0, \\ 0, & \zeta \geq 0. \end{cases} \tag{4}$$

is the Indicator function. Note that the Indicator function simply returns 1 when its argument is negative and 0 otherwise. $I\left(g\left(\mathbf{x}\right)\right)$ then indicates if g is negative at $\mathbf{x}$, i.e. if failure occurs for $\mathbf{x}$.

From the definition of expected value (see Ross [2010]) we then conclude that the probability of failure can also be written as

$$P_f = \int_\Omega I\left(g\left(\mathbf{x}\right)\right) f_X\left(\mathbf{x}\right) d\mathbf{x} = E\left[I\left(g\left(\mathbf{X}\right)\right)\right], \tag{5}$$

where $E[\cdot]$ represents the expected value of its argument. In other words, the probability of failure can also be defined as the expected value of $I\left(g\left(\mathbf{X}\right)\right)$.

4 Time-dependent Reliability Analysis

The reliability analysis problem from Eq. (5) is important for a wide variety of engineering applications. However, when time-dependent effects are considered, the problem is usually restated in order to explicitly account for them. This type of problem is known as Time-Dependent Reliability Analysis. Some examples of time-dependent problems include reliability analysis considering structures subject to ageing effects, such as corrosion; structures subject to loads with seasonal variation, such as storms; among others.

4.1 Stochastic Processes

In the context of Structural Safety, uncertain time-dependent effects are generally modeled as Stochastic Processes. A Stochastic Process can be understood as a time-dependent random variable $X(t)$, with time-dependent density function $f_X(x,t)$[2]. In Figure 1 we present an example of one realization of a given stochastic process $X(t)$. A given realization of $X(t)$ is also known as a "path" and is represented as $x(t)$. Note that a path is a time-dependent function. If one tries to observe a stochastic process several times, different paths would be observed. A group of several paths of a given stochastic process is called an "ensemble" of $X(t)$.

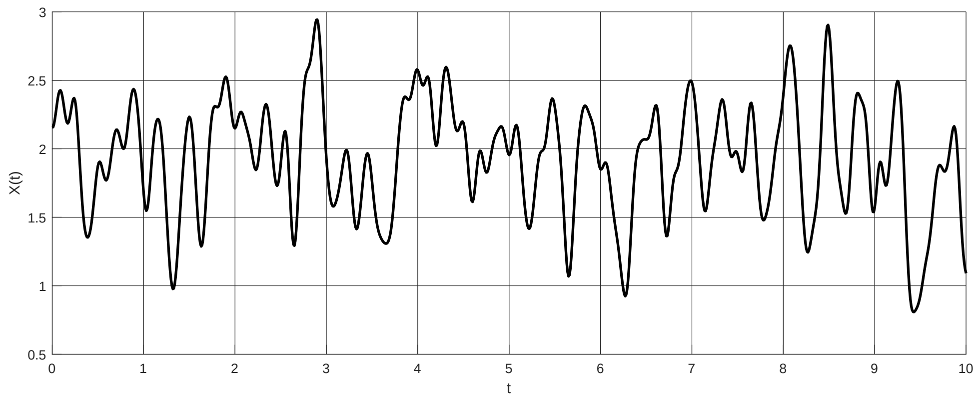

Figure 1: A path of a given stochastic process $X(t)$

Strictly speaking, in order to define a stochastic process $X(t)$ we need to define the time-dependent density $f(x,t)$ for each time instant t [Shriryaev, 1995, Loève, 1977]. This can be done explicitly or implicitly. However, explicit definition of $f(x,t)$ is uncommon, for practical reasons.

A common approach for defining a stochastic process is to define it implicitly (i.e. indirectly), by defining the following properties: a parametric distribution for $X(t)$ (e.g. Normal, Uniform, Gamma, among others); a time-dependent expected value

$$\mu(t) = E\left[X(t)\right]; \tag{6}$$

a time-dependent standard deviation

$$\sigma(t) = \sqrt{V\left[X(t)\right]} = \sqrt{E\left[(X(t) - \mu(t))^2\right]}, \tag{7}$$

where $V[\cdot]$ represents the variance, and the auto-correlation

$$\rho\left[X(t_a), X(t_b)\right] = \frac{C\left[X(t_a), X(t_b)\right]}{\sigma(t_a)\sigma(t_b)} = \frac{E\left[(X(t_a) - \mu(t_a))\left(X(t_b) - \mu(t_b)\right)\right]}{\sigma(t_a)\sigma(t_b)}, \tag{8}$$

where $C\left[X(t_a), X(t_b)\right]$ represents the covariance between $X(t_a)$ and $X(t_b)$. Note that $\rho\left[X(t_a), X(t_b)\right]$ measures the correlation between $X(t)$ at time-instants t_a and t_b.

[2]Higher order density functions are required to describe the correlation structure of the process.

If the density $f(x,t)$ of the stochastic process $X(t)$ does not depend on t the process is called "strictly stationary". In this case the process has constant expected value

$$\mu = E\left[X(t)\right] \tag{9}$$

and constant standard deviation

$$\sigma = \sqrt{V\left[X(t)\right]}. \tag{10}$$

Besides, the auto-correlation of a strictly stationary process depends only on the distance between the time instants and thus

$$\rho\left[X(0), X(\tau)\right] = \rho\left[X(t), X(t+\tau)\right]. \tag{11}$$

For this reason, the auto-correlation of a strictly stationary process is frequently written as

$$\rho(\tau) = \rho\left[X(0), X(\tau)\right] = \frac{E\left[\left(X(0)-\mu\right)\left(X(\tau)-\mu\right)\right]}{\sigma^2}, \tag{12}$$

that results

$$\rho(\tau) = \frac{E\left[X(0)X(\tau)\right]-\mu^2}{\sigma^2}. \tag{13}$$

A stochastic process that satisfy Eqs. (9), (10) and (13) is called "weakly stationary" or, more often, stationary. Note that a process $X(t)$ that satisfies Eqs. (9), (10) and (13) does not necessarily has time-independent density $f(x,t)$. In other words, a stationary process is not necessarily "strictly stationary". However, any "strictly stationary" process is also "weakly stationary". This means that "strict stationarity" is a stronger requirement.

It is also important to highlight that for some specific stochastic process, satisfaction of Eqs. (9), (10) and (13) imply "strict stationarity". This is the case of the important family of Normal/Gaussian stochastic process, where $X(t)$ has Normal/Gaussian distribution [Shriryaev, 1995, Loève, 1977]. Thus, stationary Normal/Gaussian stochastic process are also "strict stationarity". However, note that this is not the general rule.

4.2 Time-dependent Probability of Failure

Consider a time-dependent limit state function $g(\mathbf{Y}, X(t), t)$, that depend on a vector of random variables $\mathbf{Y}$, a stochastic process $X(t)$ and time t. In this case the total (or cumulative) probability of failure in the time interval $[t_1, t_2]$ can be defined as

$$P_f(t_1,t_2) = P\left[\exists\xi \in [t_1,t_2], \mathbf{y} \in \Omega | g(\mathbf{y}, X(\xi), \xi) < 0\right], \tag{14}$$

that is the probability of existing some time instant ξ in the time interval and some $\mathbf{y}$ in the support of the random variables for which failure occurs.

On the other hand, the instantaneous probability of failure is defined as

$$p_f(t) = P\left[g(\mathbf{Y}, X(t), t) < 0\right]. \tag{15}$$

Note that evaluation of the instantaneous probability of failure is basically a time-independent reliability analysis problem, since the time instant t is fixed.

In order to simplify notation a little, from now on we will represent the above quantities omitting the explicit dependence on the random vector $\mathbf{Y}$ and time t, as is commonly done in literature. In this case we write simply

$$g(X(t)) = g(\mathbf{Y}, X(t), t) \tag{16}$$

and consequently

$$P_f(t_1, t_2) = P\left[\exists \xi \in [t_1, t_2] | g(X(\xi)) < 0\right], \tag{17}$$

$$p_f(t) = P\left[g(X(t)) < 0\right], \tag{18}$$

where dependence on $\mathbf{Y}$ and t is implicit, i.e. it is implied it should be considered, but it is not written out in the expression.

4.3 Approaches for Time-dependent Reliability Analysis

There exist two main approaches for estimating the cumulative probability of failure $P_f(t_1, t_2)$. The first approach is to generate an ensemble for the stochastic process and then employ a sample-based estimate, such as Monte Carlo Simulation (MCS), that directly evaluates $P_f(t_1, t_2)$ from Eq. (17). In this case, the probability of failure is estimated from the number of cases where g becomes negative at least once in the time interval divided by the total number of cases considered. We will cover this approach in more details later this chapter.

The second approach is based on the fact that (see Ditlevsen and Madsen [1996b] for a proof and more extensive discussion on this subject)

$$P_f(t_1, t_2) \leq p_f(t_1) + \int_{t_1}^{t_2} v^+(t)dt, \tag{19}$$

where $v^+(t)$ is the so-called "up-crossing rate". Formally, the up-crossing rate is defined as

$$v^+(t) = \lim_{\Delta t \to 0} \frac{P\left[g(X(t+\Delta t)) < 0 | g(X(t)) \geq 0\right]}{\Delta t}. \tag{20}$$

Note that the up-crossing rate represents the time-instant probability of getting $g(X(t+\Delta t)) < 0$ if we have $g(X(t)) \geq 0$. In other words, the up-crossing rate can be understood as the probability of departing for the safety and getting into failure domain.

Note that integration of the up-crossing rate produces an upper bound for the the cumulative probability of failure. However, it turns out that this upper bound becomes sharp (i.e. accurate) for small $v^+(t)$ [Ditlevsen and Madsen, 1996b]. In other words, for most practical situations we can assume the approximation

$$P_f(t_1, t_2) \approx p_f(t_1) + \int_{t_1}^{t_2} v^+(t)dt, \tag{21}$$

when $v^+(t)$ is small. For this reason, several approaches estimate the cumulative probability of failure using the above approximation.

Some methods based on the integration of the up-crossing rate are PHI2 [Andrieud-Renaud et al., 2004] , PHI2+ [Sudret, 2008], JUR/FORM [Hu and Du, 2013a], TRPD [Jiang et al., 2014], iTRPD [Jiang et al., 2018] and NEWREL [Gong and Frangopol, 2019], among others. The difference between these methods concerns mainly how the up-crossing rate $v^+(t)$ is evaluated. In fact, very efficient approaches can be conceived by employment of FORM (First Order Reliability Method) [Madsen et al., 1986, Ditlevsen and Madsen, 1996b, Melchers and Beck, 2018] concepts and similar reasoning.

The time-dependent probability of failure can also be estimated using sampling-based schemes. This approach will be covered in detail in the next section.

5 Monte Carlo Simulation

5.1 Time-independent Reliability Analysis

The probability of failure from Eq. (5) can be estimated by MCS as [Madsen et al., 1986, Melchers and Beck, 2018]:

$$P_f \approx \tilde{P}_f = \frac{1}{N}\sum_{i=1}^{N} I(g(x_i)) = \frac{n_f}{N}, \tag{22}$$

where N is the sample size and n_f is the number of failures observed in the sample. In other words, the probability of failure can be estimated as the ratio between the number of failures observed and the sample size.

The variance and the coefficient of variation of the estimate $\tilde{p}_f$ are given by [Madsen et al., 1986, Melchers and Beck, 2018]

$$Var[\tilde{P}_f] = \frac{\tilde{P}_f(1-\tilde{P}_f)}{N} \tag{23}$$

$$CV[\tilde{P}_f] = \frac{\sqrt{Var[\tilde{P}_f]}}{E[\tilde{P}_f]} = \frac{1}{\sqrt{N}}\sqrt{\frac{1-\tilde{P}_f}{\tilde{P}_f}}. \tag{24}$$

The coefficient of variation $CV[\tilde{P}_f]$ is a measure of the relative error of the estimate. Note that this relative error depends on the probability of failure and the sample size N. In particular, small probabilities of failure will lead to large errors unless large samples are employed.

5.2 Time-dependent Reliability Analysis

The time-dependent probability of failure from Eq. (17) can be estimated with MCS as

$$P_f(t_1,t_2) \approx \tilde{P}_f(t_1,t_2) = \frac{1}{N}\sum_{i=1}^{N} I(g^*(x_i(t))) \tag{25}$$

with

$$g^*(x_i(t)) = \min_{t\in[t_1,t_2]} g(x_i(t)). \tag{26}$$

Note that in this case the indicator function considers the minimum value of the limit state function g in the time interval $[t_1,t_2]$, for a given realization $x_i(t)$ of the stochastic process. This is equivalent to checking if failure occurs for the worst situation observed in the time interval $[t_1,t_2]$.

However, in order to estimate the probability of failure we first need to obtain an ensemble for the stochastic process, composed by several possible paths $x_1(t), x_2(t), ..., x_N(t)$. Two approaches that can be employed for this purpose will be described in the next sections.

6 Expansion of the Optimal Linear Estimation (EOLE)

In the context of time-dependent reliability analysis, the most popular approach for obtaining an ensemble for the stochastic process is based on the EOLE (Expansion of the Optimal Linear Expansion), proposed by Li and Kiureghian [1993]. The EOLE is an expansion technique, i.e., the stochastic process $X(t)$ is represented as a series expansion. Assuming the time interval of analysis is meshed into N_t time instants $\{t_1, t_2, ..., t_{N_t}\}$, the EOLE representation of $X(t)$ is written as

$$\widehat{X}(t) = \mu(t) + \sum_{i=1}^{N} \frac{\xi_i}{\sqrt{\lambda_i}} \boldsymbol{\phi}_i^T \boldsymbol{\Sigma}(t), \tag{27}$$

where $\mu(t)$ is the expected value of $X(t)$, ξ_i are independent standard Normal random variables, $\boldsymbol{\phi}_i$ and λ_i are the eigenvectors and eigenvalues of the autocovariance matrix $\boldsymbol{\Sigma}$, with components

$$\Sigma_{ij} = Cov\left[X(t_i), X(t_j)\right] \tag{28}$$

and $\boldsymbol{\Sigma}(t)$ is a vector with components

$$\Sigma(t)_i = Cov\left[X(t), X(t_i)\right]. \tag{29}$$

Note that $\boldsymbol{\Sigma}$ is a matrix of covariances among time instants $\{t_1, t_2, ..., t_{N_t}\}$, while $\boldsymbol{\Sigma}(t)$ is a vector of covariances between time instant t and time instants $\{t_1, t_2, ..., t_{N_t}\}$. Also note that we can write

$$\Sigma_{ij} = \sigma(t_i)\sigma(t_j)\rho\left[X(t_i), X(t_j)\right] \tag{30}$$

and

$$\Sigma(t)_i = \sigma(t)\sigma(t_i)\rho\left[X(t), X(t_i)\right], \tag{31}$$

where $\sigma(t)$ is the standard deviation of $X(t)$.

As occurs in expansion schemes, the number of terms of the series can be truncated. Thus, in general we take

$$EOLE : \widehat{X}(t) = \mu(t) + \sum_{i=1}^{r} \frac{\xi_i}{\sqrt{\lambda_i}} \boldsymbol{\phi}_i^T \boldsymbol{\Sigma}(t), \tag{32}$$

where $r \leq N$ is the number of terms kept in the expansion.

From Eq. (32) we observe that the stochastic process is represented as a series of eigenvectors/eigenvalues and independent standard normal random variables ξ_i. Thus, by sampling values for the random variables ξ_i it is possible to generate several paths for the stochastic process, i.e. to generate an ensemble for $X(t)$. This is an important characteristic of EOLE (and other expansion techniques such as Karhunen-Loève), the stochastic process is written as a series of random variables. This allows sampling of the stochastic process by sampling the random variables with standard schemes.

6.1 Numerical Adjustments

In practice, the autocovariance matrix $\boldsymbol{\Sigma}$ can become ill conditioned, particularly when N_t increases. When this happens, the eigenvalues λ_i evaluated numerically can become negative, blocking the EOLE algorithm from running (note that EOLE equations assume that all eigenvalues are non-negative, otherwise we get imaginary values). In practice this means that the EOLE equations only hold when $\boldsymbol{\Sigma}$ is positive semi-definite, i.e. when all eigenvalues satisfy $\lambda_i \geq 0$. It turns out that for numerical reasons $\boldsymbol{\Sigma}$ may become indefinite in practice, i.e. there may exist some negative eigenvalues arising from numerical errors.

In order to avoid this issue, here we consider a relaxed autocovariance matrix

$$\hat{\mathbf{\Sigma}} = \mathbf{\Sigma} + \varepsilon \mathbf{I}, \tag{33}$$

where $\varepsilon > 0$ is a small relaxation parameter and $\mathbf{I}$ is the identity matrix. By summing non-negative ε to the diagonal of $\mathbf{\Sigma}$ we ensure that it will be positive definite. This is the procedure employed in this chapter.

7 Auto-Regressive (AR) Model

A stochastic process $X(t)$ can also be represented using Time Series Models. The most common models are: Auto-Regressive (AR), Moving Average (MA), Auto-Regressive with Moving Average (ARMA) and Auto-Regressive Integrated Moving Average (ARIMA). Note that the AR, the MA and the ARMA models are appropriate to represent stationary stochastic processes, while ARIMA can be used to model non-stationary processes. See Cryer and Chan [2008], Box and Jenkins [1978] for a detailed discussion on Time Series.

Consider a stationary stochastic process $X(t)$ with expected value μ. We first define

$$Y(t) = X(t) - \mu, \tag{34}$$

i.e., $Y(t)$ is the deviation from the expected value. Assuming the time interval of analysis is meshed into N_t time instants $\{t_1, t_2, ..., t_{N_t}\}$ uniformly spaced with

$$\Delta t = t_i - t_{i-1}. \tag{35}$$

The AR model can be written as

$$\widehat{Y}_i = \varphi_1 \widehat{Y}_{i-1} + \varphi_2 \widehat{Y}_{i-2} + \cdots + \varphi_p \widehat{Y}_{i-p} + \epsilon_i, \tag{36}$$

where $\widehat{Y}_i = Y(t_i)$, $\{\varphi_1, \varphi_2, ..., \varphi_p\}$ are the parameters of the model, p is the order of the model and ϵ_i is a gaussian white noise with null expected value and standard deviation σ_ϵ. Note that the AR model is a linear combination of the past p values of Y plus a white noise. This is the reason why the model is called Auto-regressive: it explicitly depends on it's own past values.

Consider the autocorrelation between time instants t, $t + n\Delta$ (i.e. the lag n autocorrelation)

$$\rho_n = \rho(n\Delta t), \tag{37}$$

where ρ is as defined in Eq. (13). It can be demonstrated that ρ_n satisfy [Cryer and Chan, 2008]

$$\begin{cases} \rho_1 = & \phi_1 & +\phi_2\rho_1 & +\phi_3\rho_2 & +... & +\phi_p\rho_{p-1} \\ \rho_2 = & \phi_1\rho_1 & +\phi_2 & +\phi_3\rho_1 & +... & +\phi_p\rho_{p-2} \\ \vdots & & & & & \\ \rho_p = & \phi_1\rho_{p-1} & +\phi_2\rho_{p-2} & +\phi_3\rho_{p-3} & +... & +\phi_p \end{cases}, \tag{38}$$

that are known as Yule-Walker equations.

The Yule-Walker equations are generally employed to predict the discrete autocorrelation of the AR model $\widehat{Y}$ as a function of the parameters ϕ_1, ϕ_2, ..., ϕ_p. Here we employ the Yule-Walker equations to obtain these parameters assuming the autocorrelation function is known. For this purpose, note that we can write the Yule-Walker equations as

$$\mathbf{R}\boldsymbol{\phi} = \mathbf{r}, \tag{39}$$

with

$$\mathbf{r} = \{\rho_1, \rho_2, ...\rho_p\}^T \tag{40}$$

$$\boldsymbol{\phi} = \{\phi_1, \phi_2, ...\phi_p\}^T \tag{41}$$

$$\mathbf{R} = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \rho_3 & \cdots & \rho_{p-1} \\ \rho_1 & 1 & \rho_1 & \rho_2 & \cdots & \rho_{p-2} \\ \rho_2 & \rho_1 & 1 & \rho_1 & \cdots & \rho_{p-2} \\ \vdots & \vdots & \vdots & 1 & \ddots & \vdots \\ \rho_{p-1} & \rho_{p-2} & \rho_{p-3} & \rho_{p-4} & \cdots & 1 \end{bmatrix}. \tag{42}$$

Besides, the components of $\mathbf{R}$ can be written as

$$R_{ij} = \rho_{j-i}, \tag{43}$$

since $\rho_0 = 1$ and $\rho_n = \rho_{-n}$.

Both $\mathbf{R}$ and $\mathbf{r}$ can be evaluated if the autocorrelation function of the stochastic process is known. In this case, by solving the system of linear equations from Eq. (39) we are able to obtain the parameters $\boldsymbol{\phi}$ of the AR(p) model directly from the autocorrelation of the stochastic process. Finally, the standard deviation of the white noise can be evaluated from [Cryer and Chan, 2008]

$$\sigma_e^2 = \sigma \left(1 - \sum_{i=1}^{p} \phi_i \rho_i \right), \tag{44}$$

where σ is the standard deviation of the stationary stochastic process.

7.1 Stationarity of Auto-Regressive (AR) Model

The AR model only results in a stationary time series if the roots of the characteristic equation

$$1 - \phi_1 x - \phi_2 x^2 - ... - \phi_p x^p = 0 \tag{45}$$

are all larger than unity in magnitude (note that some roots may be imaginary) [Cryer and Chan, 2008]. Dividing by $-\phi_p$ we can write the above equation as

$$-\frac{1}{\phi_p} + \frac{\phi_1}{\phi_p} x + \frac{\phi_2}{\phi_p} x^2 + ... + x^p = 0. \tag{46}$$

The roots of the above polynomial equation can be obtained from the eigenvalues of the companion matrix [Tismenetsky, 1992]

$$\hat{\mathbf{C}}_L = \begin{bmatrix} 0 & 0 & ... & 0 & 1/\phi_p \\ 1 & 0 & ... & 0 & -\phi_1/\phi_p \\ 0 & 1 & ... & 0 & -\phi_2/\phi_p \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & ... & 1 & -\phi_{p-1}/\phi_p \end{bmatrix}. \tag{47}$$

Thus, the AR model will be stationary if

$$|\lambda_i| > 1, i = 1, 2, ..., p, \tag{48}$$

where λ_i are the eigenvalues of

$$\hat{\mathbf{C}}_L \mathbf{v} = \lambda \mathbf{v}. \tag{49}$$

Tismenetsky [1992] demonstrates that the condition from Eq. (48) is satisfied if $\mathbf{R}$ is positive definite. It is known that the correlation matrix is always positive semi-definite, otherwise negative variance should occur. However, $\mathbf{R}$ is a Toeplitz matrix that can become singular when its order is increased. Thus, in order to ensure that the Yule-Walker equations always consider a positive definite matrix some numerical adjustment can be employed.

In order to ensure stationarity of the AR models, in this work we obtain the coefficients of the AR models from the relaxed system of linear equations

$$(\mathbf{R} + \varepsilon \mathbf{I})\, \boldsymbol{\phi} = \mathbf{r}, \tag{50}$$

where $\varepsilon > 0$ is a small relaxation parameter and $\mathbf{I}$ is the identity matrix. By summing non-negative ε to the diagonal of $\mathbf{R}$ we ensure that it will be positive definite. Thus, the resulting AR model will be stationary.

It is interesting to note that both the AR models and the EOLE expansion require some kind of relaxation in order to be numerically robust in practice. Besides, in both cases we observe that the relaxation employed must ensure positive definiteness of the autocovariance/autocorrelation matrix. Here we do not pursue this interesting idea further.

8 Numerical Examples

We now present two numerical examples in order to compare the results obtained using EOLE and the AR model to represent the stochastic processes in the context of time-dependent reliability analysis. In the examples, the autocorrelation function is assumed to be of the form

$$\rho(\tau) = \exp\left[-\left(\frac{\Delta t}{\alpha}\right)^2\right], \tag{51}$$

where α is known as the correlation length. The autocorrelation function from Eq. (51) was chosen because it is widely employed in the literature in the context of time-dependent reliability analysis.

The relative difference between the results obtained with the AR model and EOLE was evaluated as

$$e(\%) = \left|\frac{\overline{P}_f - P_{REF}}{P_{REF}}\right| 100\%, \tag{52}$$

where P_{REF} is a reference value for the probability of failure and $\overline{P}_f$ an estimate.

In both the EOLE and the AR approaches we employed a relaxation parameter equal to $\varepsilon = 10^{-12}$, that was observed to make the algorithms numerically stable without affecting accuracy of the results. It is highlighted that if no relaxation parameter is employed, both EOLE and AR become unstable if the order of the models increases too much.

8.1 Example 1: Constant Barrier

In the first example we consider a stationary Gaussian process $X(t)$ with expected value $\mu = 2$ and standard deviation $\sigma = 0.4$. The autocorrelation is given by Eq. (51) with correlation length $\alpha = 1.0$. The limit state function is given by

$$g = 3 - X(t), \tag{53}$$

i.e. failure is assumed to occur if $X(t) > 3$. The cumulative probability of failure was evaluated for the time interval $[0, 10]$, using a discretization $\Delta t = 0.1$ (i.e. 101 uniformly spaced time instants).

The reference solution $P_f = 0.099799$ was obtained using EOLE with all terms in the expansion (i.e. $r = 101$ in Eq. (32)) and a sample size $N = 10^6$. The results obtained using AR and EOLE models of different orders and a sample size $N = 10^5$ are presented in Tables 1 and 2.

Table 1: Results obtained with AR models (Example 1)

AR	P_f	$e(\%)$
$p = 2$	0.08604	0.13786711289692
$p = 4$	0.09261	0.07203478992775
$p = 8$	0.09809	0.01712442008437
$p = 16$	0.09819	0.01612240603613
$p = 32$	0.10027	0.00471948616719
$p = 64$	0.10025	0.00451908335754
$p = 100$	0.09942	0.00379763324281

Table 2: Results obtained with EOLE models (Example 1)

EOLE	P_f	$e(\%)$
$r = 2$	0.00252	0.97474924598442
$r = 4$	0.01856	0.81402619264722
$r = 8$	0.06916	0.30700708423932
$r = 16$	0.09842	0.01381777372518
$r = 32$	0.10040	0.00602210442990
$r = 64$	0.10062	0.00822653533602
$r = 100$	0.10023	0.00431868054790

The relative differences to the reference solution $e(\%)$ versus the order of the models employed is plotted in Figure 2 in log-log scale. We observe that EOLE gives very poor results with less than 8 terms in the expansion. The AR, on the other hand, is able to obtain consistent results even with models of order $p = 2$ and 4. As the order of the model is increased, both the EOLE and the AR converge to similar values.

8.2 Example 2: Beam Subject to Corrosion

In this example, adapted from Andrieud-Renaud et al. [2004], we consider a steel beam subject to corrosion. The limit state function is taken as

$$g = \frac{(b_0 - 2\kappa t)(h_0 - 2\kappa t)^2 \sigma_y}{4} - \left(\frac{F(t)L}{4} + \frac{\rho_{st} b_0 h_0 L^2}{8} \right), \tag{54}$$

where b_0 and h_0 are the initial cross section dimensions, L is the beam length, κ is the corrosion rate t is the time measured in years, σ_y is the yielding stress, ρ_{st} is the density of the material. It is assumed that the corrosion depth evolves in time as

$$d_c(t) = \kappa t. \tag{55}$$

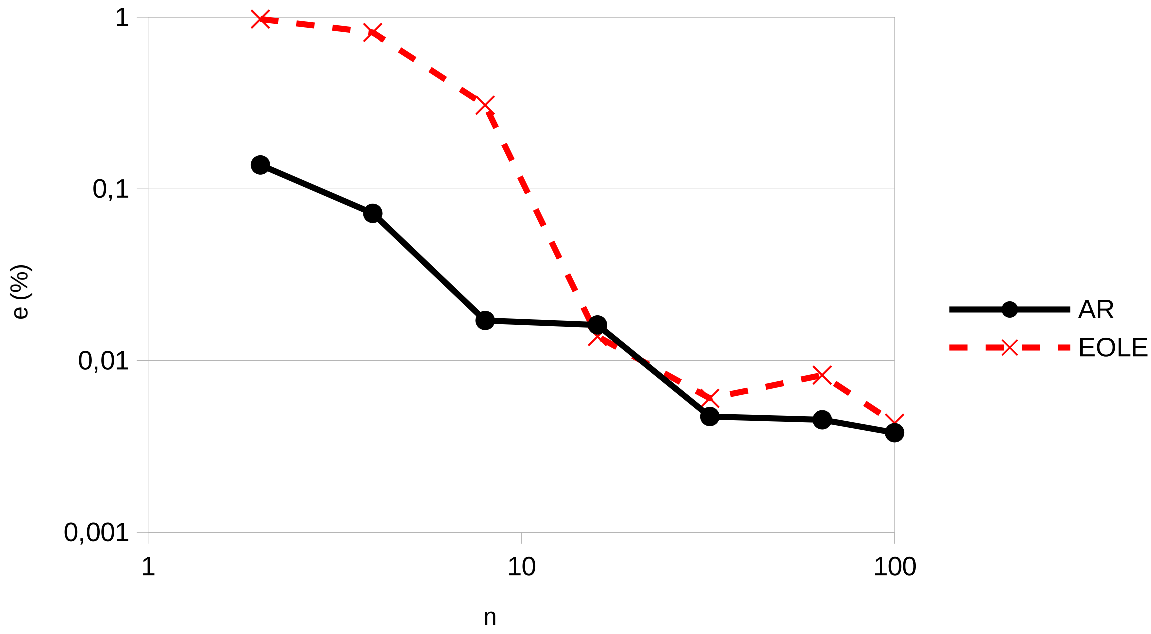

Figure 2: Convergence of AR and EOLE models (Example 1)

The random variables are as defined in Table 3. The correlation length of the Stochastic Process (i.e. applied force) is $\alpha = 1/12$ years.

Table 3: Parameters of Example 2

Parameter	Distribution	Expected value	CV*
Applied force F (kN)	Gaussian Stochastic Process	5000	20%
Yielding stress σ_y (MPa)	Lognormal	240	10%
Initial cross section width b_0 (m)	Lognormal	0.20	5%
Initial cross section height h_0(m)	Lognormal	0.04	10%
Corrosion rate κ (mm/ano)	Deterministic	0.03	-
Beam length L (m)	Deterministic	5.00	-
Steel density ρ_{st} (kN/m^3)	Deterministic	78.50	-

* CV: coefficient of variation

The time interval of analysis was taken as $[0, 20]$ years. This interval was divided into 200 uniformly spaced sub-intervals with $\Delta t = 0.1$ years. The reference solution $P_f = 0.022138$ was obtained using EOLE with all terms in the expansion (i.e. $r = 201$ in Eq. (32)) and a sample size $N = 10^6$. The results obtained using AR and EOLE models of different orders and a sample size $N = 10^5$ are presented in Tables 4 and 5.

The relative differences to the reference solution $e(\%)$ versus the order of the models employed is plotted in Figure 3 in log-log scale. We observe that EOLE gives very poor results unless almost all terms are included in the expansion (i.e. $r = 200$). The AR, on the other hand, is able to obtain consistent results even with models of order $p = 2$ and 4. For this reason, it is not possible to identify advantages of employing high order AR models in this example.

Table 4: Results obtained with AR models (Example 2)

AR	P_f	$e(\%)$
$p = 2$	0.02178	0.01617128918601
$p = 4$	0.02172	0.01888156111663
$p = 8$	0.02197	0.00758876140572
$p = 16$	0.02242	0.01273827807390
$p = 32$	0.02119	0.04282229650374
$p = 64$	0.02115	0.04462914445749
$p = 128$	0.02123	0.04101544855000
$p = 200$	0.02185	0.01300930526696

Table 5: Results obtained with EOLE models (Example 2)

EOLE	P_f	$e(\%)$
$p = 2$	2.60E-04	0.98825548830066
$p = 4$	2.30E-04	0.98961062426596
$p = 8$	4.00E-04	0.98193152046255
$p = 16$	7.50E-04	0.96612160086728
$p = 32$	0.00177	0.92004697804679
$p = 64$	0.00534	0.75878579817508
$p = 128$	0.01385	0.37437889601590
$p = 200$	0.02188	0.01165416930165

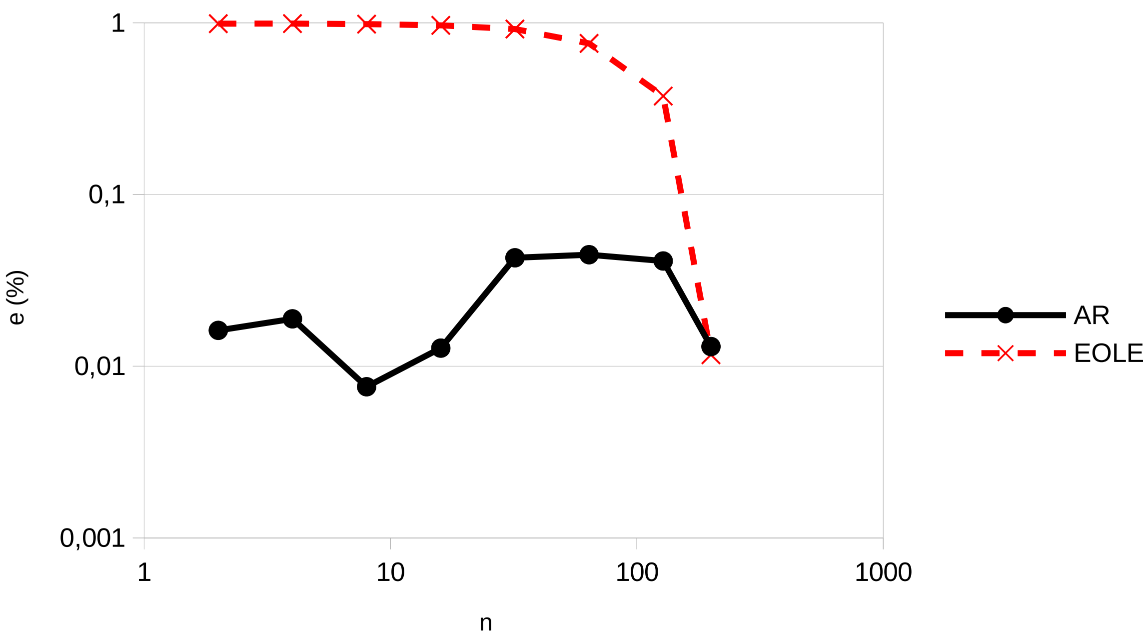

Figure 3: Convergence of AR and EOLE models (Example 2)

9 Concluding Remarks

We presented a conceptual and computational comparison between EOLE (Expansion Optimal Linear Estimation) and AR (Auto-Regressive) models to represent stochastic processes in the context of time-dependent reliability analysis. Although EOLE is by far the most widespread technique to represent stochastic process in this context, it can be applied only if the properties of the stochastic process are explicitly known. However, such information is rarely available in practice. Times Series Models such as AR, on the other hand, are efficient to represent existing data obtained from real time monitoring or available historical databases. For this reason, in this chapter we investigated if it is possible to obtain similar results using EOLE and AR models in the context of time-dependent reliability analysis.

The results obtained here demonstrated that EOLE and AR models are able to provide similar results for time-dependent reliability analysis. This is an important conclusion, since AR (and other Time Series Models) are widely employed in practice. However, some questions should be further investigated in future works. First, it is necessary to investigate if the conclusions obtained here also hold when non-stationary stochastic processes are considered. Besides, it is also interesting to study which Time Series Models (AR, ARMA, ARIMA, among others) are more fit for reliability analysis problems.

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