Chapter 18

Random Field Discretization in Stochastic Finite Elements: Element Size Effects and Error Indicators in Reliability Structural Analysis

Chapter details

Chapter DOI:

https://doi.org/10.4322/978-65-86503-88-3.c18

Chapter suggested citation / reference style:

Jorge, Ariosto B., et al. (2022). "Random Field Discretization in Stochastic Finite Elements: Element Size Effects and Error Indicators in Reliability Structural Analysis". *In* Jorge, Ariosto B., et al. (Eds.) *Uncertainty Modeling: Fundamental Concepts and Models*, Vol. III, UnB, Brasilia, DF, Brazil, pp. 613–639. Book series in Discrete Models, Inverse Methods, & Uncertainty Modeling in Structural Integrity.

P.S.: DOI may be included at the end of citation, for completeness.

Book details

Book: Uncertainty Modeling: Fundamental Concepts and Models

Edited by: Jorge, Ariosto B., Anflor, Carla T. M., Gomes, Guilherme F., & Carneiro, Sergio H. S.

Volume III of Book Series in:

Discrete Models, Inverse Methods, & Uncertainty Modeling in Structural Integrity

Published by: UnB City: Brasilia, DF, Brazil Year: 2022

DOI: https://doi.org/10.4322/978-65-86503-88-3

Random Field Discretization in Stochastic Finite Elements: Element Size Effects and Error Indicators in Reliability Structural Analysis

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Abstract

Stochastic finite element methods are used for structural reliability analysis. The element properties and parameters in element equations may involve randomness. One extra dimension is added to the deterministic finite element method to account for the variability in the parameters. Discretization of the structure can be performed and the distributed parameters can be modeled as random fields. Different methods of discretization of the random fields are reviewed in this work. Depending on the discretization method, and on the particular problem being studied, the element size will influence the accuracy and numerical stability of the results. A correlation model is assumed for the random variables that represent the random field in the discretized problem. The correlation models adopted in this work are the triangular and the exponential ones. These models depend on specific parameters: the scale of fluctuation (θ/L) and the correlation length (λ). The type of correlation model and the value of these parameters also influence the reliability index results. Depending on the correlation model adopted, a simple error indicator for the reliability index (β) can be used, by taking, for a specific mesh, the difference between the solution for two different discretization models, one that is known to overestimate the covariance matrix, and another that underestimates this matrix. For the numerical example, the reliability index results showed that both error indicators pointed out to the same range of values, for the parameter in the discretization model, in which bigger errors are expected, or in which the reliability index practically does not change. Values for the correlation model parameter that minimize the differences in the reliability index, for the various discretization models and meshes, are convenient, because, for these values, the model would be more robust and less dependent on external factors, such as the element size of the random field.

Keywords: SFEM - Stochastic Finite Element Methods, random fields, structural reliability, error estimators.

1 Stochastic Finite Elements: Introduction

Stochastic Finite Elements are structural models represented by finite elements, the properties of which involve randomness [Schueller,, 1997]. These properties are parameters or variables in the element equations. Examples are the stiffness matrix in static analysis, shape imperfection in buckling analysis, damping and mass properties in structural dynamics. One extra dimension is added to the deterministic finite element method (FEM) to account for the variability in the parameters.

Two categories of Stochastic Finite Elements procedures can be distinguished [Schueller,, 1997].The first category is the representation of Stochastic Finite Elements and their global assemblage as random structural matrices. Several representations or methods of discretization have been developed to describe spatial random fluctuations within the element. Some of these representations are the midpoint method [Hisada and Nakagiri,, 1985], the nodal point method [Hisada and Nakagiri,, 1981], the interpolation method [Liu et al.,, 1986], the local average method [Vanmarcke and Grigoriu,, 1983] and the weighted integral method [Takada,, 1990, Deodatis,, 1991]. As a tendency [Schueller,, 1997], the mid-point methods leads to an overestimation of the variance of the response, the local average method to an underestimation and the weighted integral method leads to the most accurate results. The random field may also be discretized by expansion into a series using basis random variables [Lawrence,, 1986], Karhunen-Loeve expansion [Ghanem and Spanos,, 1991, Spanos and Ghanem,, 1988], or Neumann expansion [Spanos and Ghanem,, 1988, Shinozuka and Deodatis,, 1988]. The discretization of the random field can be different from the geometric discretization of the deterministic FEM. Depending on the discretization method, and on the particular problem being studied, the element size will influence the accuracy and might lead to singular matrices and numerical instability of the results.

The second category is the evaluation of the stochastic response of the Finite Elements model due to its randomness. The most widely used procedure [Schueller,, 1997] for evaluating the stochastic response is the perturbation approach [Hisada and Nakagiri,, 1985, Liu et al.,, 1986, Haldar,, 1997]. It is well adapted to the finite element formulation and capable to evaluate first and second moment properties of the response in an efficient manner, for small deviation from the center value. For an accurate reliability assessment, importance sampling and the response surface method might be employed, where the perturbation procedure might be utilized to estimate the response in the neighborhood of some specified center value (design point). Perturbation algorithm is used in this way for the establishment of the sampling distribution in an adaptive procedure.

2 Random Field Discretization Aspects: Introduction

The choice of the random field discretization and of the solution refinement approaches depend on various aspects.The first aspect is what is the problem to be solved and what stochastic finite element formulation will be used. Section 4 describes some existing formulations. The choice will have to consider if the problem is linear or nonlinear, if it is static or time-dependent, if it has or not a closed-form solution, what are the random fields to be considered, if some hypothesis can be done, like assuming Gaussian variables or assuming known covariances or assuming known moments of the distributions. The choice of the formulation may depend also on what information is wanted as a response of the system, and on the available CPU-time to solve the problem.

The second aspect is what kind of discretization will be done for the random fields and to what extent mesh refinement can be done. Section 3 describes some representations of random fields and some existing discretization methods. Depending on the stochastic finite element formulation being used, discretization may be intrinsic to the method, like when using numerical integration. Depending on the assumed or known types of random variables, certain types of discretization may

fit better, or give more accurate results, or demand less CPU-time. Depending on the discretization method adopted, some parameters may be influent in the problem, like the correlation length, or give rise to numerical instabilities, leading to upper and lower bounds for the mesh element size.

A third aspect that may be considered is the influence of the random fields in the response of the system. A sensitivity analysis may be performed [Mahadevan and Haldar,, 1991, Mahadevan,, 1997] to evaluate this influence. Section 5 describes the sensitivity analysis approach. Depending on the problem, a sensitivity index may be defined so that the relative influence of the various random fields in the response of the system can be assessed. If this index is very small compared to the other random fields, the corresponding random field could be approximated by a single random variable, without much loss in accuracy. Further, if this sensitivity index is very low – less than a cutoff value, say 0.05 – the stochasticity of the random field could be simply ignored, and the field treated as a deterministic variable. This sensitivity analysis may be dependent on the first two aspects, the SFEM formulation and the random field discretization method adopted. If the assumptions behind a discretization method for one random field are not reasonably satisfied, results may be inaccurate, and that may include also the sensitivity index. This means that a decision to disregard a random field could be done based in inaccurate information, and maybe with a different discretization method this decision would not be made.

3 Representation of Random Fields

3.1 Stochastic discretization

This review of the different discretization procedures follows Matthies [Matthies et al.,, 1997]. In deterministic finite elements, continuous functions are represented by a system of equations written in terms of a finite set of nodal point parameters. In stochastic finite elements, random fields or random processes are represented in terms of a set of discrete random variables. Their statistical properties are expected to depend on the chosen finite element mesh. The discretization of the random fields may be chosen independently of the one for the deterministic part. The selection criteria for the mesh of the random fields and of the geometry are different.

3.2 Element size and mesh refinement in random field discretization

The discretization of the random fields depends on three factors [Liu,, 1990]:

The first factor is accuracy. In view of accuracy, the element size is controlled by a parameter, the correlation length, that describes the rate of fluctuation of the random field. The correlation length of a random field is a measure of the distance over which the correlation coefficient approaches a small value. Hence, smaller random field elements should be used if the correlation length is large. If this correlation length is infinity, the random field is, in fact, a single random variable. If the correlation length is small, the element size has to be short enough so that a good approximation of the random field by the random variable in within the element is obtained. According to Liu, [1990], an element size of one half of the correlation length suffices for a satisfactory representation of the random field. Another parameter used byHisada and Nakagiri, [1981] is λ_u, the largest wave number that cannot be disregarded in the power spectral density $S_{HH}(\lambda)$. They concluded that the nodal interval should be taken to be less than $0.25\lambda_u$. Any of these parameters provide an upper bound for the distance between two adjacent random variables.

The second factor is the numerical stability of the transformation to the standard normal space. An excessively fine mesh yields highly correlated random variables. Therefore, the correlation matrix is nearly singular and, in cases where a transformation to the standard normal space is required, this may cause numerical difficulties. Hence, this factor provides a lower bound on the element size. In the local average method [Vanmarcke and Grigoriu,, 1983], it was observed that

the reduction of the point variance under local averaging was inversely proportional to T, the averaging interval, for large values of T. This proportionality constant, the scale of fluctuation θ, is the approximate length over which strong correlation persists in the random field. Element sizes can be chosen larger than the scale of fluctuation to avoid numerical difficulties.

The third factor is the efficiency of the computational method. A smaller number of basic variables would require less computation time, especially when partial derivatives of the functions are to be computed. Thus, the elements should be as large as possible from this viewpoint. Matrix condensation [Mahadevan,, 1997] can be done to reduce the computational effort in deterministic analysis as well as in the computation of response sensitivity.

These controlling factors are different from the factor in the selection of the finite element mesh, which is basically governed, in the static problem, by the gradient in the displacement field.

Der Kiureghian and Ke, [1988] suggested separate meshes for the finite element and for the random fields. In general, it is appropriate to use a finite element mesh that satisfies the requirements based on the displacement gradient and the correlation lengths of all random fields, and then for each random field choose a mesh that is coincident with or coarser than the finite element mesh such that the corresponding probability transformation remains stable. In their work, the influence of the size of the random field elements in the accuracy and stability of the results was not explored.

Extra difficulties in investigating the effect of element size on the stability and accuracy of the results may arise due to numerical errors dependent on the solution algorithm or in problems where no exact solution is available. This is the case in reliability analysis, which does not incorporate random fields. Hence, there is no exact solution that can be used to check the accuracy of the analysis for a certain random field mesh, and only comparisons between results of a coarse mesh and a refined mesh can be done.

An analysis of the error resulting from discretizing random fields in the SFEM was presented by Spanos and Zeldin, [1997]. This error analysis is performed for the shift invariant subspace discretization method which encompasses the midpoint method, the interpolation method and the local averaging method, giving a simple criterion of estimating the accuracy of these random field discretization schemes. Their work investigates the influence of the random field discretization on the response variability of the stochastic mechanics problem, deriving an a-posteriori bound for the response variability.

The kind of discretization method affects the quality and the properties of the representation. For different random fields in a structural problem, optimal mesh sizes might be different. This meshes might be different from the mesh of the deterministic problem. And for each random field this optimal mesh size may also vary depending on the type of discretization method being used. Upper and lower bounds for the element size for a certain random field may exist and be different from one discretization method to another.

3.3 Point discretization methods

3.3.1 General

Point discretization methods represent the random field $H(\mathbf{x})$ by its values at one or more points. The random field evaluated at a certain point is a random variable. At point i the discretized value is given by

$$H_i = H(\mathbf{x}_i) \tag{1}$$

The mean, the variance and the marginal distribution of the random variable $H(\mathbf{x}_i)$ correspond to those of the random field at the point $\mathbf{x}_i$. Covariance (matrix) between two points is computed in

terms of a given autocovariance function $C_{HH}(\mathbf{x}, \mathbf{y})$:

$$C_{H_i H_j} = C_{HH}(\mathbf{x}_i, \mathbf{x}_j) \tag{2}$$

All point discretization methods have advantages and disadvantages.
The advantages are:

i) the covariance matrix can be easily computed;

ii) the covariance matrix is positive definite;

iii) same distribution function in both the discretized and in the continuous case. Therefore, there is no restriction of this method to Gaussian random fields only.

The disadvantages are:

i) The mesh size has to be small compared to the correlation length, so that random properties could be considered as constant within the element;

ii) The shape and size of all the elements should be the same;

iii) A fine mesh requires much CPU-time. A coarse mesh will not fulfill the accuracy requirements. These methods will only fulfill accuracy requirements with coarse meshes for medium to long correlation distances.

3.3.2 The midpoint method

The value at the center point of the element is used to represent the random field. The coordinates of the centroid are given in terms of the nodal coordinates $\mathbf{x}_j^{(i)}$ of the i^{th} element:

$$\mathbf{x}_i^* = \frac{1}{N} \sum_{j=1}^{N} \mathbf{x}_j^{(i)} \tag{3}$$

where N denotes the number of nodes of the element. The midpoint method tends to over-represent the variability of the random field within each element.

3.3.3 The nodal point method

In this method, the coordinates of the points used to represent the random field are given by

$$\mathbf{x}_i^* = \mathbf{x}_i \tag{4}$$

where $\mathbf{x}_i$ denotes the coordinate vector of the i^{th} global node.

3.3.4 The integration point method

If numerical integration is applied to the problem, then the random field is implicitly discretized by means of the integration points. This discretization is not done in the first step, but when integration is needed. An example is the evaluation of covariances and correlations when using local averages. The coordinates of the points depend on the rules of the integration scheme adopted. A common scheme is the Gauss quadrature. Also, this method has been used directly by Brenner, [1993].

3.4 The local averaging method

The local averaging method [Vanmarcke and Grigoriu,, 1983] provides discretized values of the random field given by

$$H_i = \frac{1}{vol(\Omega_i)} \int_{\Omega_i} H(\mathbf{x}) d\mathbf{x} \tag{5}$$

where Ω_i is the domain over which the integration has to be performed. The domain represents an averaging time, in the case of a stochastic process, or the area of the i^{th} element in the case of stochastic finite elements. If only functions of space are involved, the method is called the spatial averaging method. The spatial averaging method tends to underrepresent the variability within each element, and yields accurate results even for rather coarse meshes [Der Kiureghian and Ke,, 1988]. This method is practically limited to cases where the field is Gaussian. For other fields, the distribution function of the random variable H_i is difficult or even impossible to obtain.

3.5 The interpolation method

Liu et al., [1986] suggested an approximation of the random field $H(\mathbf{x})$ by q nodal values using shape functions $N_i(\mathbf{x})$ by

$$H(\mathbf{x}) = \sum_{i=1}^{q} N_i(\mathbf{x}) H_i \tag{6}$$

where H_i are the nodal values of $H(\mathbf{x})$ at suitable nodes $\mathbf{x}_i$. The shape functions need not to be the same from the finite element shape functions. Nodal points need not to coincide with the ones of finite elements. The expected value and the variance of the random field written with this discretization are, respectively:

$$E[H(\mathbf{x})] = \sum_{i=1}^{q} N_i(\mathbf{x}) E[H_i] \tag{7}$$

$$Var[H(\mathbf{x})] = \sum_{i,j=1}^{q} N_i(\mathbf{x}) N_j(\mathbf{x}) C_{HH}(\mathbf{x}_i, \mathbf{x}_j) \tag{8}$$

where $C_{HH}(\mathbf{x}, \mathbf{y})$ is the auto-covariance function of the random field. Depending on the shape functions, good approximations are to be expected even for coarse meshes and elements of quite different size. Choosing the nodes to be the centroids of the random field elements and the shape functions to be unity inside each element and zero elsewhere, reduces the interpolation method to the midpoint method.

3.6 The Kriging technique

The Kriging technique [Der Kiureghian et al.,, 1991] (optimum linear estimation method) can be seen as the linear regression of the original random field onto some linear functionals of the field, usually simple [Ditlevesen,, 1995], which make use of a discretizing mesh different from the one from the deterministic part. The covariance function of the random field is then employed for an interpolation. If the stochastic and deterministic meshes become identical, the method reduces to the weighted integral method Deodatis, [1991]. This method is valid for Gaussian random fields only, because this is one of the assumptions in linear regression.

3.7 Series expansion method

3.7.1 General

In the above methods the defining parameters of the discretization were linear functionals of the random field. These methods are convenient if the covariance function of the random field is known and if the random field itself is Gaussian. If either the random field is not Gaussian (thus, the random field is modeled as a nonlinear function of a Gaussian field) or the deterministic quantity of interest is a nonlinear function of other random variables (hence, the covariance function is

not easy to compute), these methods are not directly applicable anymore. Nonlinear methods such as the homogeneous chaos expansion method [Ghanem and Spanos,, 1991] and the perturbation method [Hisada and Nakagiri,, 1985, Liu et al.,, 1986] may be used in this case.

3.7.2 The basis random variable method

Lawrence, [1986, 1987] represents the random field $H(\mathbf{x})$, of which the first and second moment functions are known, by a series

$$H(\mathbf{x}) = \sum_{j=1}^{\infty} h_{0j}\phi_j(\mathbf{x}) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} h_{ij}e_i\phi_j(\mathbf{x}) \tag{9}$$

where the $\phi_j(\mathbf{x})$ are a set of linearly independent and preferably orthogonal deterministic functions satisfying all the kinematic boundary conditions. The statistically independent basis random variables e_i have the properties

$$E[e_i] = 0 \quad for \quad i = 1, 2, \cdots, \tag{10}$$

$$E[e_i e_j] = \delta_{ij} \tag{11}$$

where δ_{ij} is the Kronecker delta. The series is truncated by using only $i = N$ basis random variables (necessary to model the properties of the random field), and $j = M$ linearly independent deterministic functions. Approximating $H(\mathbf{x})$ over the interval $[a, b]$, the coefficients h_{ij} can be determined by a least-square fit of the first and second moments of the truncated random field to the first and second moments of $H(\mathbf{x})$.

3.7.3 The kernel expansion method

The kernel expansion method, also known as Karhunen-Loeve expansion [Ghanem and Spanos,, 1991, Spanos and Ghanem,, 1988] is a type of spectral representation of the random field $H(\mathbf{x}, \theta)$ by

$$H(\mathbf{x}, \theta) = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\xi_i(\theta)\phi_i(\mathbf{x}) \tag{12}$$

where $\xi_i(\theta)$ denotes a denumerable set of random variables to be determined, λ_i is some constant and $\phi_i(\mathbf{x})$ are orthonormal deterministic functions which can be related to the covariance kernel of $H(\mathbf{x}, \theta)$. The symbol θ denotes an element of the space of random events. After truncating the series at N terms, the random field is expressed by

$$H(\mathbf{x}, \theta) = \overline{H}(\mathbf{x}) + \sum_{i=1}^{N} \sqrt{\lambda_i}\xi_i(\theta)\phi_i(\mathbf{x}) \tag{13}$$

where $\overline{H}(\mathbf{x})$ is the expected value of the random field and λ_i and $\phi_i(\mathbf{x})$ are the eigenvalues and eigenfunctions of the covariance kernel, respectively, obtained as the solutions to the eigenvalue problem:

$$\int_{\Omega} C(\mathbf{x}_1, \mathbf{x}_2)\phi_i(\mathbf{x}_2)d\mathbf{x}_2 = \lambda_i\phi_i(\mathbf{x}_1) \tag{14}$$

where $C(\mathbf{x}_1, \mathbf{x}_2)$ is the auto-covariance function of the stochastic field. The random variables $\xi_i(\theta)$ have properties similar to those of the basis random variables:

$$E[\xi_i(\theta)] = 0 \tag{15}$$

$$E[\xi_i(\theta)\xi_j(\theta)] = \delta_{ij} \tag{16}$$

This expansion minimizes the mean-square error introduced by the truncation. For a decreasing correlation length the number of terms has to grow for a certain required accuracy Ghanem and Spanos, [1991], Spanos and Ghanem, [1988]. If the covariance function of the field to be expanded is not known another kind of spectral representation can be done, the homogeneous chaos expansion Ghanem and Red-Horse, [1999], Ghanem, [1999].

3.7.4 The homogeneous chaos expansion method

In this method the solution field can be expressed as

$$u = h[\xi_i(\theta), \mathbf{x}], \tag{17}$$

where u denotes the solution field and $h[.\,]$ is a nonlinear functional of its arguments. First, the random fields are replaced by their corresponding Karhunen-Loeve expansions. Next, a nonlinear expansion of $h[.\,]$ is done using the set of all orthogonal polynomials Γ_p not exceeding order p, called the polynomial chaos of order p, in terms of a set of orthogonal, uncorrelated, Gaussian random variables $\{\xi_i(\theta)\}_{i=1}^{\infty}$. Assuming symmetry of the polynomials, which is always possible [Ghanem and Spanos,, 1991], the random variables $\xi(\theta)$ of the random field, whose statistical properties are not known, can be expressed as

$$\begin{aligned}\chi(\theta) = a_0\Gamma_0 + \sum_{i_1=1}^{\infty} a_{i_1}\Gamma_1(\xi_{i_1}(\theta)) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} a_{i_1 i_2}\Gamma_2(\xi_{i_1}(\theta),\xi_{i_2}(\theta)) \\ + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} a_{i_1 i_2 i_3}\Gamma_3(\xi_{i_1}(\theta),\xi_{i_2}(\theta),\xi_{i_3}(\theta)) \\ + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2}\sum_{i_4=1}^{i_3} a_{i_1 i_2 i_3 i_4}\Gamma_4(\xi_{i_1}(\theta),\xi_{i_2}(\theta),\xi_{i_3}(\theta),\xi_{i_4}(\theta)) + \cdots\end{aligned} \tag{18}$$

where Γ_p are successive polynomial chaoses of their arguments and $a_{i_1,\cdots,i_p}$ are the deterministic constants. The random variable series, truncating after the J^{th} polynomial reads

$$\chi(\theta) = \sum_{i_1=1}^{J} \kappa_i \Psi_i[\{\xi_r\}] \tag{19}$$

where κ_i and $\Psi_i[\{\xi_r\}]$ are identical to $a_{i_1,\cdots,i_p}$ and $\Gamma_p(\xi_{i_1},\cdots,\xi_{i_p})$, respectively. A computational implementation of the method uses a finite set of random variables, the n-dimensional polynomial chaos of order p, which is a subset out of the polynomial chaos of order p, function of only n of the random variables ξ_i. The convergence properties of the representation depend on the choice of the ξ_i out of the infinite set as well as on the number of random variables. This expansion is restricted to multidimensional functions of Gaussian variables. For non-Gaussian fields, an approximation via the polynomial chaos could be used.

3.7.5 The perturbation method

A Taylor series representation [Ghanem and Spanos,, 1991] of a random field $H(\mathbf{x})$ about the mean value of the m random variables $\{h_i(\mathbf{x})\}_{i=1}^m$ yields

$$\begin{aligned} H(\mathbf{x}) &= g(h_1(\mathbf{x}), \cdots, h_m(\mathbf{x})) \\ &= g(\overline{h}_1(\mathbf{x}), \cdots, \overline{h}_m(\mathbf{x})) + \sum_{i=1}^{m} \frac{\partial g(\overline{h}_1(\mathbf{x}), \cdots, \overline{h}_m(\mathbf{x}))}{\partial h_i(\mathbf{x})} [h_i(\mathbf{x}) - \overline{h}_i(\mathbf{x})] \\ &+ \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\partial^2 g(\overline{h}_1(\mathbf{x}), \cdots, \overline{h}_m(\mathbf{x}))}{\partial h_i(\mathbf{x}) \partial h_j(\mathbf{x})} [h_i(\mathbf{x}) - \overline{h}_i(\mathbf{x})][h_j(\mathbf{x}) - \overline{h}_j(\mathbf{x})] + \cdots \\ &= \overline{g} + \sum_{i=1}^{m} \overline{g}_i \epsilon_i + \sum_{i=1}^{m}\sum_{j=1}^{m} \overline{g}_{ij} \epsilon_i \epsilon_j + \cdots \end{aligned} \tag{20}$$

Evaluating all random fields and functions involved in the equation leads to a multidimensional polynomial in ϵ. Equating polynomials of the same order on both sides yields a set of equations to be solved. This method is well suited for small fluctuations (in comparison with the mean value), in which a first-order perturbation analysis show good results. Second-order perturbation analysis improves first-order results only little and may require a huge amount of CPU-time. The perturbation method requires many terms for representing random fields with a large coefficient of variation. But the use of higher order terms of the expansion may induce numerical rounding error even for small coefficients of variation, affecting the accuracy of the representation [Ghanem and Spanos,, 1991].

4 Stochastic Finite Element Formulations

This review of the different Stochastic finite element formulations follows Matthies et al., [1997]. The general matrix form of the equation of motion is

$$\mathbf{M}\ddot{\mathbf{v}}(t) + \mathbf{C}\dot{\mathbf{v}}(t) + \mathbf{K}\mathbf{v}(t) = \mathbf{f}(t) \tag{21}$$

with proper initial conditions. The matrices $\mathbf{K}$, $\mathbf{C}$ and $\mathbf{M}$ and the force vector $\mathbf{f}$ may involve randomness. In the case of static problems, this equation reduces to

$$\mathbf{K}\mathbf{v} = \mathbf{f} \tag{22}$$

The stiffness matrix can be split into two parts, the non-fluctuating part $\mathbf{K}_0$ and the fluctuating part $\mathbf{K}_*$. To solve the static problem, several methods were proposed, as follows.

4.1 Stochastic global stiffness matrix expansions

4.1.1 The weighted integral method

The weighted integral method [Takada,, 1990, Deodatis,, 1991] does not require a separate discretization of the random field. There is no independent mesh for the stochastic part, but everything is determined by the chosen mesh for the deterministic part. The stochastic field can be written as

$$\alpha(\mathbf{x}) = \alpha_0(\mathbf{x}) \cdot [1 + f_*(\mathbf{x})] \tag{23}$$

where $f_*(\mathbf{x})$ is a zero mean homogeneous isotropic Gaussian field with auto-covariance function C_{ff} and $\alpha_0(\mathbf{x})$ is the mean of the random field at $\mathbf{x}$. The fluctuating part of the element stiffened

matrix can be written as

$$\mathbf{K}_*^{(e)}(i,j) = \mathbf{K}_{*,ij}^{(e)} = a_{ij} \int_{\Omega_e} x^r y^s z^t \alpha_0(x,y,z) dx dy dz \tag{24}$$

where the weighted integral is a random variable, $X_i^{(e)}$, and the associate weight $a_{ij} = \Delta\mathbf{K}^{(e)}$ is a deterministic constant matrix. Polynomial shape functions are used. The number of weighted integrals N_w depends on the type of element used, which determines the shape function to use. The random field is projected onto the space spanned by the shape functions. The fluctuating part for element stiffness matrix is

$$\mathbf{K}_*^{(e)} = \sum_{i=1}^{N_w} X_i^{(e)} \Delta\mathbf{K}_i^{(e)} \tag{25}$$

and the random global stiffness matrix is

$$\mathbf{K} = \sum_{e=1}^{M_e} \mathbf{K}^{(e)} = \sum_{e=1}^{M_e} (\mathbf{K}_0^{(e)} + \sum_{i=1}^{N_w} X_i^{(e)} \Delta\mathbf{K}_i^{(e)}) \tag{26}$$

The stochastic part of the stiffness matrix is described in terms of $N_w \cdot M_e$ weighted integrals. The response statistics can be determined by a first-order Taylor series expansion of the response vector $\mathbf{v}$ about E, where the mean values of the random variables are $\overline{X}_i^{(e)}$

$$\mathbf{v} \approx \mathbf{v}_0 + \sum_{e=1}^{M_e} \sum_{i=1}^{N_w} \frac{\partial \mathbf{v}}{\partial X_i^{(e)}} \Bigg|_E (X_i^{(e)} - \overline{X}_i^{(e)}) \tag{27}$$

Where $\frac{\partial \mathbf{v}}{\partial X_i^{(e)}} \Big|_E$ is obtained by partially differentiating Eq. 22

$$\frac{\partial \mathbf{v}}{\partial X_i^{(e)}} \Bigg|_E = -\mathbf{K}_0^{-1} \frac{\partial \mathbf{K}}{\partial X_i^{(e)}} \Bigg|_E \mathbf{v}_0 \tag{28}$$

From Eq. 27 the statistics $E[\mathbf{v}]$ and $Cov[\mathbf{v},\mathbf{v}]$ can be obtained [Matthies et al., 1997]. The major drawback of this method is the first order Taylor series expansion which partially ruins the improvement in quality of representation of the method. This method yields good results in the domain of short correlation distances, where the midpoint discretization method fails. For correlation distance about equal or larger than the geometric size of the problem, both methods approach each other. If numerical integration is applied, instead of analytical integration, then there is an implicit discretization done by the integration points used in the quadrature.

4.1.2 The Neumann series expansion

In conjunction with stochastic finite elements, the expansion of the inverse of the stochastic global stiffness matrix in a Neumann series can be done [Ghanem and Spanos,, 1991, Spanos and Ghanem,, 1988, Shinozuka and Deodatis,, 1988]. The stochastic static problem Eq.22 can be written

$$\mathbf{v} = (\mathbf{K}_0 + \mathbf{K}_*)^{-1}\mathbf{f} = (\mathbf{I} + \mathbf{K}_0^{-1}\mathbf{K}_*)^{-1}\mathbf{K}_0^{-1}\mathbf{f} = (\mathbf{I} + \mathbf{J})^{-1}\mathbf{K}_0^{-1}\mathbf{f} \tag{29}$$

The term in parentheses can be expanded in a kind of geometric series, called the Neumann series

$$\mathbf{K}^{-1} = (\mathbf{K}_0 + \mathbf{K}_*)^{-1} = (\mathbf{I} - \mathbf{J} + \mathbf{J}^2 - \mathbf{J}^3 + \cdots)\mathbf{K}_0^{-1} = \sum_{k=0}^{\infty} (-\mathbf{K}_0^{-1}\mathbf{K}_*)^k \mathbf{K}_0^{-1} \tag{30}$$

This expansion has the advantage that only the non-fluctuating part of the stiffness matrix has to be inverted and this only once. The response vector is now represented by the series

$$\mathbf{v} = (\mathbf{I} - \mathbf{J} + \mathbf{J}^2 - \mathbf{J}^3 + \cdots)\mathbf{v}_0 = \mathbf{v}_0 - \mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 + \cdots \tag{31}$$

This series solution is equivalent to the recursive formula

$$\mathbf{K}_0\mathbf{v}_i = \mathbf{K}_*\mathbf{v}_{i-1} \quad i = 1, 2, \cdots \tag{32}$$

Once $\mathbf{v}_0$ is obtained from $\mathbf{v}_0 = \mathbf{K}_0^{-1}\mathbf{f}$, every subsequent $\mathbf{v}_i$ is determined. Shinozuka and Deodatis, [1988] derived analytical formulas for the response variability using first-order Neumann expansion, $\mathbf{v} = (\mathbf{I} - \mathbf{J})\mathbf{v}_0$. The improved Neumann expansion [Ghanem and Spanos,, 1991] is obtained when the Karhunen-Loeve expansion is used to represent the stochastic field describing the randomness of the stiffness matrix and then Neumann series expansion of the response vector is written. The number of terms included in both Karhunen-Loeve and Neumann expansions will dictate the cost in CPU-time and the accuracy of the model.

4.1.3 The Taylor series expansion

The perturbation method [Hisada and Nakagiri,, 1985, Liu et al.,, 1986] can be applied on the stochastic stiffness matrix, where the random field $\alpha(x, y, z) = H(\mathbf{x})$ has to be discretized by a set of N zero mean random variables $\{\alpha_i\}_{i=1}^N$, as follows:

$$\mathbf{K} = \mathbf{K}_0 + \sum_{i=1}^{N} \mathbf{K}_i^I \alpha_i + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \mathbf{K}_{ij}^{II} \alpha_i\alpha_j + \cdots \tag{33}$$

where $\mathbf{K}_0$ is the stiffness matrix evaluated at the mean value of the vector of random variables $\alpha = \mathbf{0}$, $\mathbf{K}_i^I = \frac{\partial \mathbf{K}}{\partial \alpha_i}\Big|_{\alpha=\mathbf{0}}$, $\mathbf{K}_{ij}^{II} = \frac{\partial^2 \mathbf{K}}{\partial \alpha_i \partial \alpha_j}\Big|_{\alpha=\mathbf{0}}$ and so on. This method can be applied to static and dynamic problems, linear and nonlinear. The response vector $\mathbf{v}$ and the external force vector $\mathbf{f}$ can be expanded in a similar way as the stiffness matrix [Matthies et al.,, 1997]. In comparison with the Neumann series expansion, the Taylor series expansion has the severe disadvantage that partial derivatives with respect to the random variables are required. Computational effort increase dramatically when including higher-order terms. The results are in general quite satisfying for small coefficients of variation of the underlying stochastic fields, using just first- and second-order perturbation methods. For a wider range of fluctuations, a successive perturbation method [Hisada and Nakagiri,, 1985] can be applied.

4.2 Monte-Carlo simulation

Monte Carlo simulation (MCS) can be used to generate samples of the global stiffness matrix and, hence, by solving the linear equation system, samples of the response can be obtained. As the generation of random variables according to a specific distribution is an easy task in the uncorrelated space, usually the covariance matrix of all random variables involved in the problem has to be calculated [Matthies et al.,, 1997]. Integrals have to be performed to compute the covariance matrix between two element stiffness matrix entries, with extensive need of computational time even for small problems. The number of random variables has to be decreased by one of the methods in Section 3, which involve a point discretization, or by the weighted integral method in Section 4.1.1. In this case, as the integrations would only be performed numerically, this method will also involve a point discretization. The total number of random variables involved in a stochastic problem is at most equal to the total number of different integration points used. The distance between two integration points has to be less than the upper bound discussed in Section 3.1.

4.3 Implicit time integration with Hermite-Gauss quadrature

This method was developed by Liu et al., [1986] and applied to a stochastic finite element system described by Eq. 21, where $\mathbf{M}$ and $\mathbf{f}(t)$ are deterministic, $\mathbf{C} = \mathbf{0}$, and $\mathbf{K}$ is a stochastic stiffness matrix involving a set of random variables $\{\alpha_i\}_{i=1}^{N}$. The system is solved using a finite difference recursive algorithm. Numerical integrations have to be performed at every time step. Accuracy and the need of CPU-time are determined by the number of integration points used and also by the observation time.

4.4 SFEM using the homogeneous chaos expansion

Homogeneous chaos expansion can be applied to stochastic finite elements [Ghanem and Spanos,, 1991]. In Eq. 22, the Karhunen-Loeve expansion is used to represent the random field involved in the global stiffness matrix. The Karhunen-Loeve expansion is then applied to the solution vector $\mathbf{v}$. As the covariance C_{vv} of the response is not known, the homogeneous chaos expansion is applied to the random variables χ_j describing the response $\mathbf{v}$. Multiplying the obtained equation by all the polynomial chaoses and requiring the mean-square representation error to be minimal leads to an equation that, after imposing the kinematic boundary conditions will supply a set of algebraic equations to be solved for the coefficients of the polynomial chaoses. Properties of these chaoses, nominally, the zero mean and the orthogonality, are then used to evaluate the mean and the covariance of the response.

4.5 Basis random variable-based SFEM

Considering a static finite element problem, the involved random functions can be replaced by a series using basis random variables as described in Section 3.7.2. The first- and second- moments of these functions are assumed to be known. The vector of random displacement functions $\mathbf{v}$ is expanded as

$$\mathbf{v} = \mathbf{N}^T\mathbf{d}_0 + \sum \mathbf{N}^T\mathbf{d}_i e_i \tag{34}$$

where $\mathbf{N}^t$ is the matrix of deterministic shape functions defined within the boundaries of the respective element, $\mathbf{d}_0$ is the mean nodal displacement vector and the $\mathbf{d}_i$ comprise the second-moment properties of the nodal displacement vector. In the case of linear elasticity, the material stiffness is a linear operator represented by the constitutive matrix $\mathbf{D}$ in $\mathbf{D}\epsilon = \sigma$. The basis random variable expression of the matrix $\mathbf{D}$ and of the nodal force $\mathbf{f}$ give

$$\mathbf{D} = \mathbf{D}_0 + \sum\sum \mathbf{D}_{ij} e_i \psi_j \tag{35}$$

$$\mathbf{f} = \mathbf{f}_0 + \sum \mathbf{f}_i e_i \tag{36}$$

where the matrix $\mathbf{D}_0$ is the mean constitutive matrix, $\mathbf{D}_{ij}$ is a deterministic coefficient matrix, where the coefficients are defined by the least-squares fit described in Section 3.7.2 and ψ_j is a matrix of deterministic shape functions defined over the entire structure. The strain-displacement relation is $\epsilon = \mathbf{B}\mathbf{v}$. The random strain energy density is $\frac{1}{2}\epsilon^T\sigma = \frac{1}{2}\epsilon^T\mathbf{D}\epsilon$. The random strain energy of the element is the integral of the random strain energy density over the volume of the element. Integrating over the volume of an element, taking the expected value and differentiating with respect to the $\mathbf{d}_i$ yields the element stiffness matrix. For the integration, the function matrix ψ_j can be evaluated at the center of the element (midpoint method) or at the Gauss points (integration point method). The assembled finite element equation system is similar to the deterministic system, with an extra set of dimensions, included due to the random character of the problem. The global stiffness matrix is sparse and can be solved by an iterative method.

5 Sensitivity Analysis

The SFEM formulation and the discretization method may require huge computational effort. In order to ensure efficiency of the computations, without loss of accuracy, some random fields related to some of the distributed parameters of the problem may be considered, as an approximation, as random variables. A sensitivity analysis may be performed [Mahadevan and Haldar,, 1991, Mahadevan,, 1997] to evaluate the influence of the random fields in the response of the system. In the sensitivity analysis approach, depending on the problem, a sensitivity index may be defined so that the relative influence of the various random fields in the response of the system can be assessed. If this index is very small compared to the same index from other random fields, the corresponding random field could be approximated by a single random variable, without much loss in accuracy of the response. Further, if this sensitivity index is very low – less than a cutoff value, say 0.05 – the stochasticity of the random field could be simply ignored, and the field treated as a deterministic variable. This sensitivity analysis may be dependent on the SFEM formulation and on the random field discretization method adopted. If the assumptions behind a discretization method for one random field are not reasonably satisfied, results may be inaccurate, and that may include also the sensitivity index. This means that a decision to disregard a random field could be done based in inaccurate information, and maybe with a different discretization method this decision would not be made.

Sensitivity analysis is not the only guideline to reduce the size of the problem without much loss in accuracy. The representation of a distributed parameter by a single random variable implies a perfectly correlated random field, i. e. , a random field with infinite correlation length. Therefore, if a distributed parameter has a very large correlation length, it may be modeled as a random variable, despite its having a high sensitivity index.

6 Error Indicators For the Random Field Discretization in Reliability Structural Analysis

The previous sections review the existing stochastic finite element formulations, the discretization methods proposed up to now and some aspects already discussed in the literature concerning element sizes and mesh refinement.

Discretization has to be addressed even for the SFEM methods that does not require it explicitly if numerical integration is to be performed.

Upper and lower bounds of the error exist only for certain discretization methods. They were obtained in the context of some specific problems, like static formulation (structures) or reliability analysis, and they may be problem-specific.

Error estimates that could lead to an adaptive remeshing of the random field exist only for the shift invariant subspace discretization method, which uses an spectral approach.

Depending on the discretization method, and on the particular problem being studied, the element size will influence the accuracy and numerical stability of the results. A correlation model is assumed for the random variables that represent the random field in the discretized problem. The correlation models adopted in this work are the triangular and the exponential ones. These models depend on specific parameters: the scale of fluctuation (θ/L) and the correlation length (λ). The type of correlation model and the value of these parameters also influence the reliability index results.

Two different error indicators for the reliability index (β) are proposed in this work. These error indicators can be useful to evaluate the influence of the random field discretization model and mesh for stochastic finite element methods (SFEM) as applied to structural reliability analysis.

1. Depending on the correlation model adopted, a simple error indicator for the reliability index (β) can be used, by taking, for a specific mesh, the difference between the solution for two different discretization models, one that is known to overestimate the covariance matrix, and another that underestimates this matrix. In this work, the methods used to over- and under-estimate the covariance matrix are the midpoint method and the local average method, respectively.

2. Another error indicator can be easily established by taking differences between two different random field meshes, for the same deterministic finite element mesh and the same discretization model.

It has to be pointed out that it is necessary to first change some characteristic of the discretization of the problem and then compare the solutions for the reliability index for both cases. In the above error indicators, either the random field discretization model is changed, or the mesh is refined. If Monte Carlo Simulation (MCS) for correlated variables were performed for a certain discretization model / mesh, it will give the same results as in $FORM$ $method$ 1, because to perform the correlated MCS, it is necessary to get the same set of uncorrelated reduced normal variables Y as in $FORM$. This means that MCS will be simulating the very same limit state equation then $FORM$, and the results for the reliability index will be the same. This means that the comparison between $FORM$ and MCS does not give any other error indicator that could be used.

7 Numerical Example

A numerical example is performed to compare the two error indicator approaches. This example is presented in Mahadevan and Haldar, [1991]. A cantilever beam made of structural steel, with Young's modulus E and moment of inertia I, has a uniform load W acting on it vertically downward. For this example, the beam is discretized in one, two and four elements. The discretization in four elements is shown if Fig. 1.

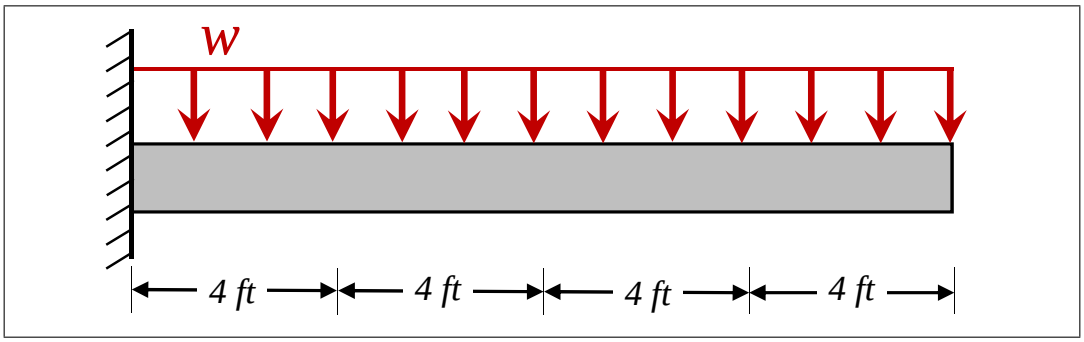


Figure 1: Numerical example: cantilever beam. Figure shows discretization with 4 elements: W1, W2, W3 & W4 - adapted from Jorge, [2004]

7.1 SFEM random field discretization for the beam

7.1.1 Beam considered as one element only

Assuming all parameters are deterministic, the solution for the vertical deflection at the free end of the beam is $y_{max} = \frac{WL^4}{8EI}$. The finite element solution for one element gives also the exact solu-

Table 1: Reliability analysis of cantilever beam: W,E and I as random variables - adapted from Jorge, [2004]

Quantity	Units	Sensitivity Index	Initial Checking point	Final checking point
W	k/in	-1.0000	0.040	0.048
E	ksi	0.0000	29000.000	28396.361
I	in^4	0.0002	301.000	296.678

β=1.053
Number of iterations = 3

tion. This means that no improvement in the deterministic solution is obtained while subsequently discretizing the beam. When considering the stochasticity off these parameters, the difference in the reliability index results for the various random field meshes of for the different discretization models is not affected by the deterministic FEM mesh. Any changes in the solution can be related to the discretization of the random fields only.

The limit state considered is that the vertical deflection at the free end of the beam is less or equal then length/200, with length in inches. Considering W, E and I as random variables, the limit state equation is

$$g = R - S = EI(y_{adm} - y_{max}) = EI\frac{L}{200} - \frac{WL^4}{8} \tag{37}$$

Assuming the statistical description of the basic variables as

$$W \sim N(0.04\frac{K}{in}; 0.008\frac{K}{in})$$
$$E \sim N(29000ksi; 1740ksi)$$
$$I \sim N(301in^3; 15.05in^3)$$

A reliability analysis $FORM\ Method$ 1 is performed as in Haldar and Mahadevan, [2000]. The results are shown in Table 1.

The sensitivity indexes in Table 1 indicated that the random variables E and I could be considered as deterministic in Eq. 37. Reliability analysis results are, in this case:

Checking point: $W = 0.049\frac{K}{in}$
$\beta = 1.166$
Iterations: 2

7.1.2 Beam discretized in two elements

The parameters W, E and I are now random fields. Let's assume E and I as deterministic parameters. Let's assume that the random field W has normal distribution at every point with same mean, but different variance as a function of the longitudinal coordinate x, for $0 \leq x \leq L$. Let's consider a linear variation of the coefficient of variation (δ) with respect to x: $\delta = 0.048 + \frac{x}{L}0.024$ so that its average is $\delta = 0.06$, which is the value that was used when considering W as a random variable, previously.

The random field W is then discretized into two elements, W_1 and W_2, while parameters E and I are considered as deterministic. A closed form for the vertical deflection at the free end of the beam y_{max} can be obtained in terms of W_1 and W_2, so that limit state equation can be written as

$$g = R - S = EI(y_{adm} - y_{max}) = EI\frac{L}{200} - \frac{L^4}{6144}(112W_1 + 656W_2) \tag{38}$$

where the random variables W_1 and W_2 are correlated. A reliability analysis with correlated variables is performed by rewriting the limit state equation with respect o a set of uncorrelated reduced normal variables Y and then using $FORM\ Method\ 1$ [Haldar and Mahadevan,, 2000].

At this point, a model for the correlation of the random variables W_1 and W_2 has to be assumed. The model expresses the decay of the variance function with the distance between two points. Two models are used in this work: the triangular and the exponential [Mahadevan and Haldar,, 1991].

In the triangular model, the approximation for the decay of the variance function is given by

$$\gamma_x(T) = 1 - \frac{T}{3\theta_x}, T < \theta_x$$
$$\gamma_x(T) = \frac{\theta_x}{T}(1 - \frac{\theta_x}{3T}), T > \theta_x \tag{39}$$

where $\gamma_x(T)$ is the variance function of the random field X, θ_x is the scale of fluctuation, and T is the distance. Using this approximation, the correlation coefficient between two discretized random variables in the i^{th} and j^{th} elements may be expressed as

$$\rho(x_i, x_j) = \frac{1}{2}\left\{ (k-1)^2\gamma_x\left[\frac{(k-1)L}{N}\right] - 2k^2\gamma_x\left[\frac{kL}{N}\right] + (k+1)^2\gamma_x\left[\frac{(k+1)L}{N}\right] \right\} \tag{40}$$

in which $k = |i - j|$.

In the exponential model, the correlation coefficient between any two random variables E_i and E_j obtained by the discretization of E is computed using

$$\rho(E_i, E_j) = exp\left[-\left(\frac{\Delta x}{\lambda_x L} \right)^2 \right] \tag{41}$$

where Δx is the difference between the x coordinates of the midpoints of the elements i and j on the random field discretization and $\lambda_x L$ is the correlation length. λ_x is a dimensionless constant that characterizes the length over which the autocorrelation function decays to a pre specified value.

For each of these correlation functions, two random field discretization models are performed: the midpoint method and the local average method. In the midpoint method, the correlation function is obtained directly from each of the correlation models Equations 40 and 41. In the local average method, the correlation function is obtained as [Spanos and Zeldin,, 1997]

$$\rho(x_i, x_j)\big|_{ave} = \frac{1}{|\Omega_i||\Omega_j|}\int_{\Omega_i}\int_{\Omega_j}\rho(x_i, x_j)dx_i dx_j \tag{42}$$

7.1.3 Beam discretized in four elements

Random field W is discretized into four elements, W_1, W_2, W_3 and W_4, while parameters E and I are considered as deterministic. A closed form for the vertical deflection at the free end of the beam y_{max} can be obtained in terms of W_1, W_2, W_3 and W_4, so that limit state equation can be written as

$$g = R - S = EI(y_{adm} - y_{max}) = EI\frac{L}{200} - \frac{L^4}{6144}(15W_1 + 97W_2 + 239W_3 + 417W_4) \tag{43}$$

where the random variables W_1 and W_2 are correlated. Similarly to the previous case, a reliability analysis with correlated variables is performed by rewriting the limit state equation with respect

o a set of uncorrelated reduced normal variables Y and then using $FORM\ Method\ 1$ [Haldar and Mahadevan,, 2000]. Triangular and exponential correlation models are performed, using the midpoint method for the random field discretization.

7.2 Reliability index and error indicator results

Different values are assumed for the parameters θ_x (scale of fluctuation) and $\lambda_x L$ (correlation length) of the triangular and exponential models, respectively. Figure **??** shows the influence of the mesh refinement when the random field is discretized using the midpoint method. The results for the reliability index are plotted for the mesh with two elements and for the mesh with four elements.

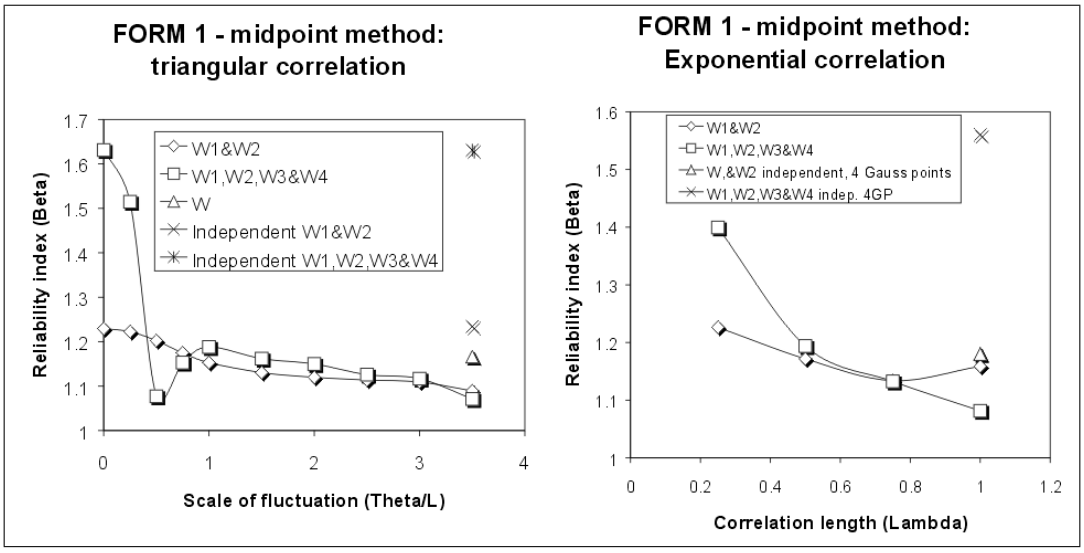


Figure 2: Influence of random field mesh refinement on reliability index for triangular and exponential correlation models (adapted from Jorge, [2004]).

Figure 3 shows the influence of the random field discretization method for a particular mesh. The results for the reliability index are plotted for the mesh with two elements for the midpoint method and for the local average method.

Figure 4 shows the error estimates obtained both from the change in the random field discretization method (for a particular mesh) and from the mesh refinement (for a particular random field discretization method). The differences in the reliability index are obtained starting from the mesh with two elements, with the midpoint method, and either changing the method to local average (first error indicator) or keeping the midpoint method and refining the mesh (second error indicator).

For the numerical example performed, the reliability index results for the error in Fig. 4 showed that both error indicators pointed out to the same range of values, for the parameter in the discretization model, in which bigger errors are expected, or in which the reliability index practically does not change. Values for the correlation model parameter that minimize the differences in the reliability index, for the various discretization models and meshes, are convenient, because, for these values, the model would be more robust and less dependent on external factors, such as the element size of the random field.

These error indicators are qualitatively comparable. although they do not shoe the same magnitude for the error, they show big errors in the same locations. To locate the range of the pa-

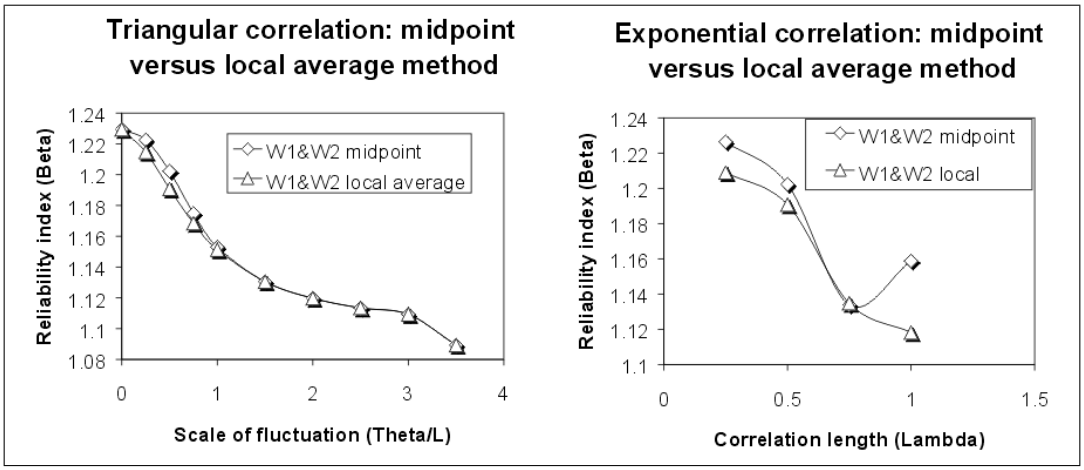


Figure 3: Influence of random field discretization method on reliability index for triangular and exponential correlation models (adapted from Jorge, [2004]).

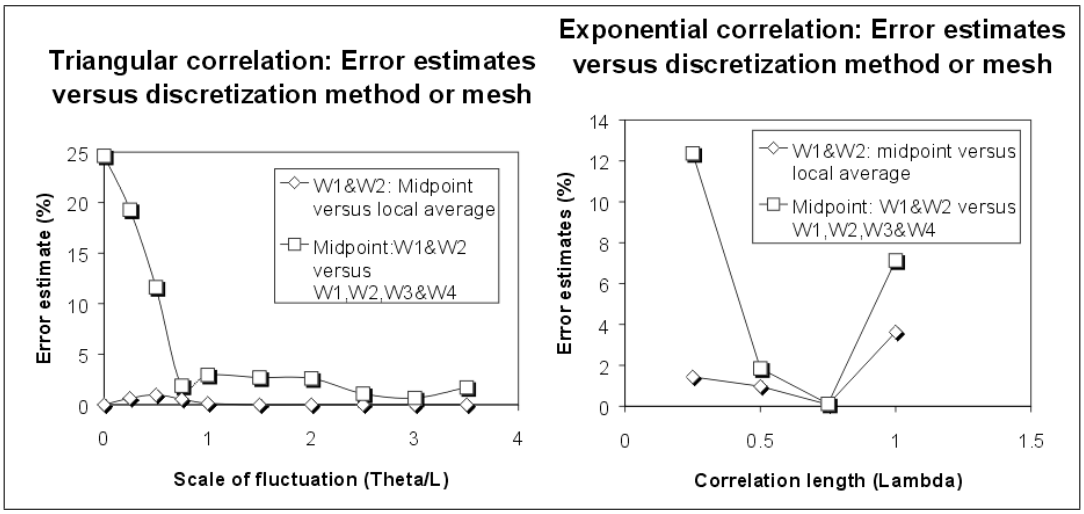


Figure 4: Error estimates (%) obtained by changing the discretization model or by random field mesh refinement (adapted from Jorge, [2004]).

rameter values in which the error is small, both methods are equivalent. In this case, changing the discretization method for a coarse mesh can be an option to its refinement. The change from midpoint method to local average method gives, with less computational effort, the range of values of the parameters of the correlation model in which this model is more robust with respect to the random field mesh refinement. There is no need to make random field mesh refinements $a-priori$ to find the appropriate values for the corresponding parameters.

8 Concluding Remarks

This work reviews the existing stochastic finite element formulations, the discretization methods proposed up to now and some aspects already discussed in the literature concerning element sizes and mesh refinement.

Discretization has to be addressed even for the SFEM methods that does not require it explicitly if numerical integration is to be performed.

Upper and lower bounds of the error exist only for certain discretization methods. They were obtained in the context of some specific problems, like static formulation (structures) or reliability analysis, and they may be problem-specific.

Error estimators for the reliability index (β) were proposed in which comparisons were done between two methods of random field discretization (for the same random field mesh) and between two different random field meshes (for the same random field discretization method). It is necessary to first change some characteristic of the discretization of the problem and then compare the solutions for the reliability index for both cases. The comparison between $FORM$ and Monte Carlo Simulation MCS for correlated variables does not give any other error indicator that could be used.

The reliability index results for the error showed that both error indicators pointed out to the same range of values, for the parameter in the discretization model, in which bigger errors are expected, or in which the reliability index practically does not change. These error indicators pointed out the values for the correlation model parameter that minimize the differences in the reliability index, for the various discretization models and meshes. These are the values of the parameters to be used for the discretization model to be more robust and less dependent on the chosen model or on the element size of the random field.

Also, as the error indicators were comparable, changing the discretization method for a coarse mesh can be an option to its refinement. The change from midpoint method to local average method gives, with less computational effort, the range of values of the parameters of the correlation model in which this model is more robust with respect to the random field mesh refinement. There is no need to make random field mesh refinements $a - priori$ to find the appropriate values for the corresponding parameters.

A safer design of a structure could include a study on the correlation parameters based on the outlined error indicator approaches, and then the evaluation of the lowest value of the reliability index for the appropriate range of the correlation parameters, which will give the biggest value of the probability of failure of the structure. The approximation of the random field by a system of random variables gives, using this approach, a conservative value for the reliability index.

9 Perspectives for Future Work in Random Field Discretization

Regarding random field discretization, there are some open fields of study, such as:

- A general study of what SFEM formulations are best adapted to a random field discretization;
- A study of what method of random field discretization is well suited to each of the SFEM formulations;
- A study on error bounds, error estimation and adaptivity for the various methods of random field discretization;
- A study on computational efficiency of random field discretization methods and remeshing approaches, including how different random field discretizations may interfere, modify or interact with the sensitivity index.

9.1 Perspectives for Application in Aerospace Structures

There are several potential applications of the random field approach discussed herein. A very good example is the field of aerospace structures, which usually involves demanding design requirements due to the challenging environmental and operational conditions. One interesting problem that has recently gained increased attention from researchers is the development of reliable damage identification procedures for structural health monitoring for aerospace structures. This is an application that involves stochasticity and uncertainties and could certainly benefit from the topics covered in the present chapter.

As an example, consider the problem of detecting possible damaged areas in a metallic honeycomb sandwich plate, such as the one illustrated in Figure 5 (a). One interesting approach is the use of variations in the modal strain energy due to the presence of damage in order to detect and locate regions with possible damage, such as delaminations, manufacturing imperfections due to flawed curing process, etc. [Domingues, 2019a,b]. The developed method relies on a numerical description of the stiffnes distribution over the panel and on the modal parameters of the pristine and potentially damaged panels. A finite element model (FEM) is built in order to generate a stiffness matrix required for the calculations. Domingues [2019b] developed detailed 3D FEM models using shell shell elements, as depicted in Figure 5 (b), and also simplified, 2D models using orthotropic plate elements, as shown in Figure 5 (c). In both cases, numerical models were adjusted using the results obtained from experimental modal analysis. Figure 5 (d) is a representation of the 9 candidate regions for damage investigation. It is worth noting that, for the small 30cm by 30cm specimen used in the investigation, even the simplified, 2D model involved over 50,000 degrees of freedom (DOF), whereas the experimental model was based on 49 DOFs from the measurement points. Therefore calculation of the damage index based on modal strain energy demands a dynamic condensation procedure.

Figure 6 presents the results of a simulated damage identification for the described reduced specimen.Damage was simulated as a 1% reduction of the elastic module in each region.The damage index was calculated for all 9 regions for each case of damage. It can be noticed that the largest calculated index value is clearly well correlated for every damage scenario, even for such a small level of damage severity, successfully detecting and locating the damaged region. However, the implementation was based on a completely deterministic approach. It is expected that the presence of uncertainties from a more realistic application will impact the methodÂ´s ability to produce reliable results. An interesting possible extension of the method is to represent the strain energy over the panel as a random field, where each region will have its strain energy described by a random variable. The variables may be considered, in a first approximation, as uncorrelated, rendering a diagonal covariance matrix. Different damage scenarios and sources of uncertainties from both the numerical and the experimental data may result in more complex cases where the covariance matrix could produce off-diagonal terms.

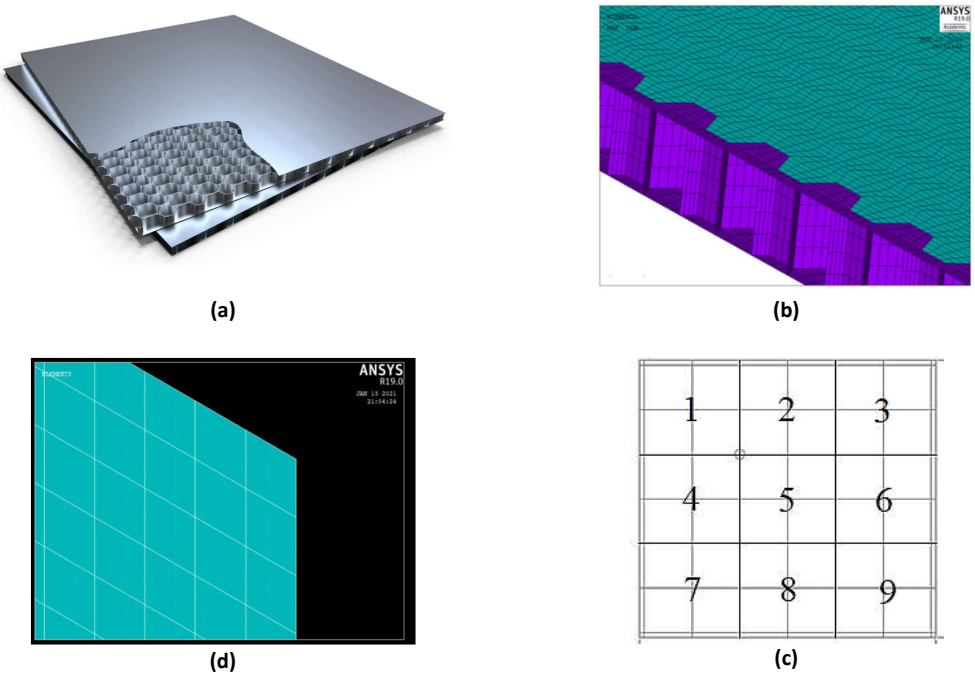


Figure 5: Models for damage detection in aerospace structures (a) Al-Al honeycomb panel for aerospace application; (b) detailed 3D FEM model (c) equivalent orthotropic 2D FEM model (d) Sub-regions for damage interrogation

9.2 Perspectives for Application in Aeronautical Composite Structures

Regarding the subject of random field, there is still a range of application possibilities, mainly in relation to composite materials applied (or not) in the aeronautical industry. Such materials have a high strength/weight ratio, however, they have greater complexity in relation to their manufacture and failure modes.

Figure 7 shows, by way of example, some suggestions and application perspectives related to the theme. Figure 7a and 7b show optimization studies of isogrid-type structures that are widely applied in the aerospace sector. As it is known that boundary conditions can present substantial variability, adopting adequate uncertainty models is imperative. Equally important, Figure 7c shows a structural (multi-objective) optimization study of a composite aircraft wing. Once again, uncertainties must be considered in the mood to obtain an effective and reliable final component. Figure Xe also shows a case of structural optimization already in the field of additive manufacturing of unconventional (auxetic) structures with negative Poisson ratio. Similarly, load variations are also present and must be duly considered.

Figure 7d also provides an example of application in the field of damage identification. This field is quite susceptible to noise (environmental, sensors, etc.). A field of study within Structural Health Monitoring concerned sensor optimization. Figures 7f and 7h show examples of sensor optimization application in aeronautical structures in order to obtain the highest quality of information (modal or static) possible. Since data are noisy in *in situ* works, once again it is necessary to use uncertainty studies for correct decision making.

Optimization under uncertainty is of paramount importance in the structural field. Figure 7g

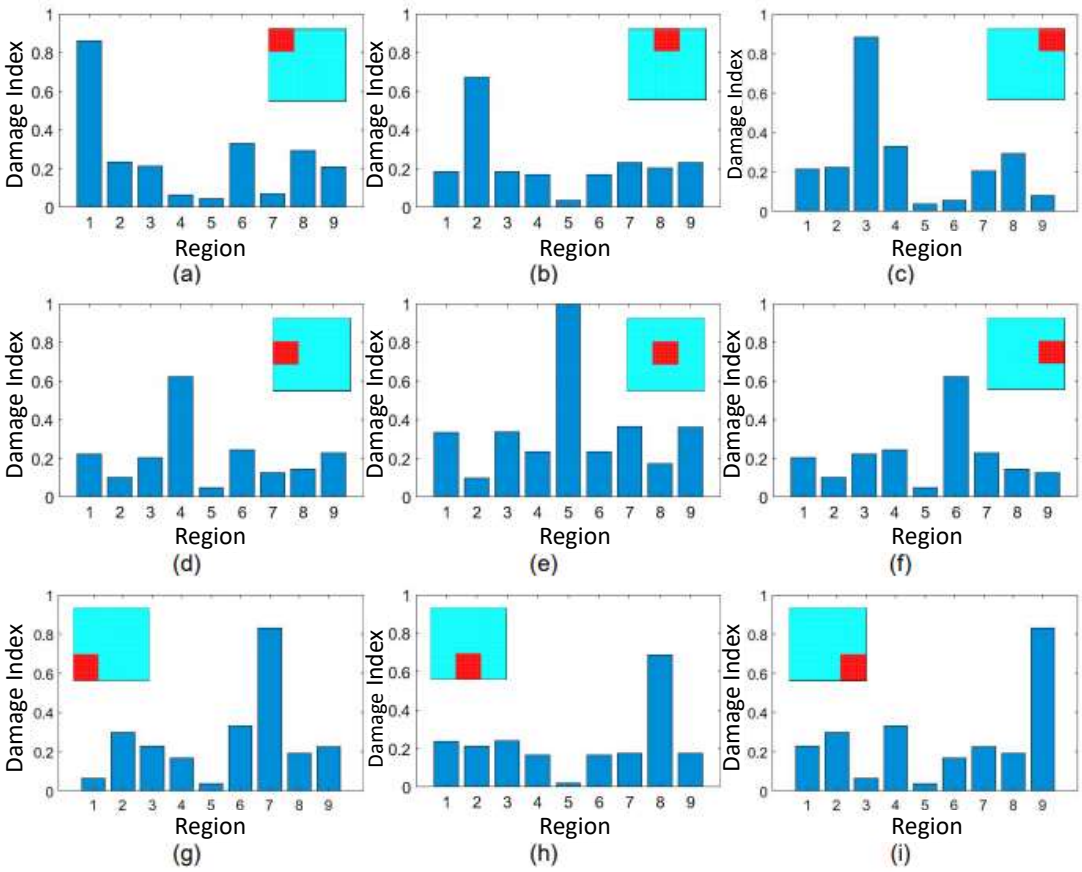
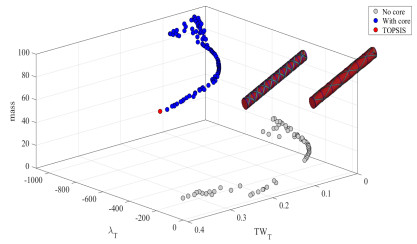


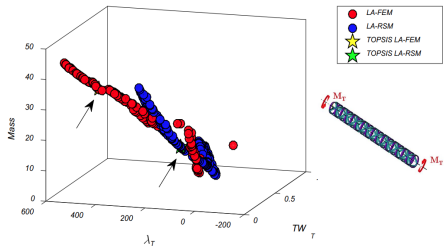
Figure 6: Modal strain energy damage index for honeycomb sandwich panel. Damage in regions (a) 1,(b) 2, (c) 3,(d) 4,(e) 5 ,(f) 6,(g) 7,(h) 8,(i) 9

shows an application perspective in optimizing the manufacture of composites with interrupt layers or simply drop-off ply.

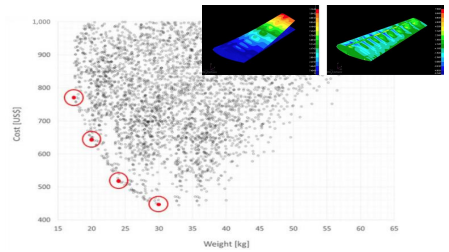
Still on the subject of damage and vibration data, Figures 7i and 7j show two different perspectives of application in methods that make use of signal processing techniques. Signal processing techniques (e.g.: wavelet, short-term Fourier transform, etc.) are vital in order to extract the greatest quantity and/or quality of information hidden in a given signal. The effectiveness of a given technique directly depends on the clarity of a signal. Therefore, properly treating a vibration signal is very important in order to obtain a robust and effective method of detecting, classifying and/or identifying damage.



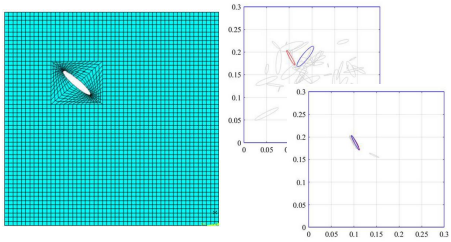
(a) Isogrid optimization (Francisco, [2021])



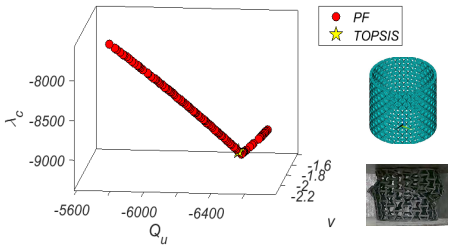
(b) Isogrid optimization (Pereira, [2022])



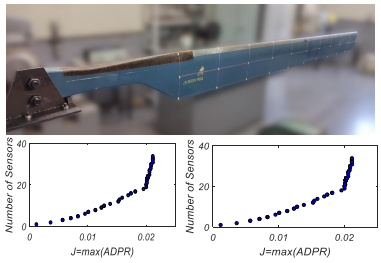
(c) Aircraft wing optimization (Ricco, [2021])



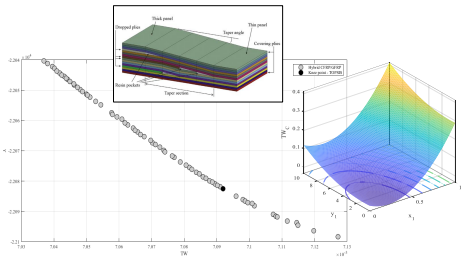
(d) Crack and delamination identification (de Assis, [2021])



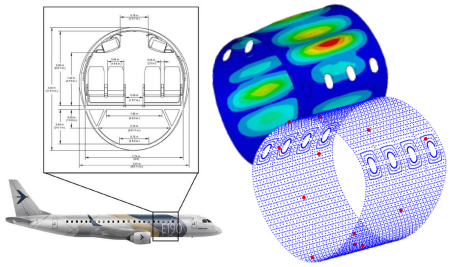
(e) Auxetic structure optimization



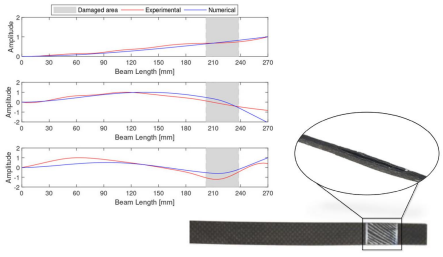
(f) Sensor placement optimization (Pereira, [2022])



(g) Ply drop-off optimization (Diniz, [2022])



(h) Sensor Placement Optimization (Gomes, [2020])



(i) Wavelet Based Damage Identification (Oliver, [2022])

Figure 7: Some examples in aeronautical composite structures.

Acknowledgements

The first author would like to acknowledge the support from the Structural Reliability Research Group at Vanderbilt University, where he was a Post-doctoral Research Associate in 2002 Structural Reliability Research Group - Faculty and Staff, [n.d.].

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