# Chapter 19

# The Boundary Element Method for Potential Problems

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# The Boundary Element Method for Potential Problems

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**Abstract**

*This chapter presents the boundary element method applied to potential problems. The integral equation is obtained for the Laplace equation and discretized into boundary elements. Constant, linear, and quadratic boundary elements are considered. The method is applied to some numerical examples and results are compared to analytical solutions. A convergence study is carried out in order to access the behaviour of the method with mesh refinement.*

## 1 Introduction

Several physical problems in nature can be modeled as a boundary value problem (BVP), in which a partial differential equation is valid in the domain being considered, some boundary conditions are prescribed in the domain boundary, and also some initial conditions may be given, for transient, non-stationary, problems.

The Finite Element Method (FEM) is a numerical method to solve a BVP by replacing the original problem by an approximate domain integral representation, obtained from a weighted residuals approach.

In the FEM equations, an auxiliary problem is introduced, from which weight functions were used in the integral representation of the problem. Also, the geometric discretization of the problem domain is followed by the representation of the unknown functions over the defined sub-domains (or, finite elements) satisfying certain continuity requirements at the boundaries of each sub-domain. An appropriate choice of the weight functions leads to the formation of a system equations comprising of symmetric matrices, assembled from the contribution of each finite element. The element equations are said to have local support, as all information required to solve these equations is limited to the geometry and quantities of interest inside the element and in its boundaries, and no information is needed from the other elements elsewhere in the domain.

The boundary element method (BEM) is another numerical method to solve a BVP, in which the original problem is replaced by an exact boundary integral representation, obtained from the application of the proper integral identities for the particular problem being studied. The resulting boundary integral representation of the original BVP relates domain integrals to boundary integrals. Unlike the finite element method, where the whole problem domain is discretized, the numerical solution scheme in this method requires discretization of the boundary of the problem, reducing the number of unknowns. For instance, in the case of a 3D domain, it is necessary to discretize the surface.

## 2 Boundary integral equations

In this section, the boundary integral equation for the Laplace problem will be developed. This equation will then be discretized into boundary elements, thus obtaining the boundary element formulation.

Given Laplace's equation

$$\nabla^2 T = 0, \tag{1}$$

multiplying the equation (1) by a weight function $\omega(x,y)$ and integrating over the domain $A$, it is assumed that the result of the integral is zero (weighted residual method). Thus, one has:

$$\begin{aligned}
\iint_A (\nabla^2 T)\omega dA &= 0, \\
\iint_A \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)\omega dA &= 0, \\
\iint_A \frac{\partial^2 T}{\partial x^2}\omega dA + \iint_A \frac{\partial^2 T}{\partial y^2}\omega dA &= 0
\end{aligned} \tag{2}$$

By the Gauss-Green theorem, we have:

$$\int_s f(x,y) n_x ds = \int_A \frac{\partial f}{\partial x} dA$$

where $f$ is a function, $n_x$ is the component in the $x$ direction of the vector $\vec{n}$ normal to the boundary $s$ of the area $A$. Applying the theorem given in the first part of Eq.(2), we have:

$$\int_s \frac{\partial T}{\partial x}\omega n_x ds = \int_A \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\omega\right) dA.$$

Applying the product of functions derivative rule, we have:

$$\int_s \frac{\partial T}{\partial x}\omega n_x ds = \int_A \frac{\partial^2 T}{\partial x^2}\omega dA + \int_A \frac{\partial T}{\partial x}\frac{\partial \omega}{\partial x} dA.$$

Rewriting the terms of the previous equation, follows:

$$\int_A \frac{\partial^2 T}{\partial x^2}\omega dA = \int_s \frac{\partial T}{\partial x}\omega n_x ds - \int_A \frac{\partial T}{\partial x}\frac{\partial \omega}{\partial x} dA. \tag{3}$$

Similarly, we obtain:

$$\int_A \frac{\partial^2 T}{\partial y^2}\omega dA = \int_s \frac{\partial T}{\partial y}\omega n_y ds - \int_A \frac{\partial T}{\partial y}\frac{\partial \omega}{\partial y} dA. \tag{4}$$

Replacing Eqs.(3) and (4) in Eq.(2), we have:

$$\int_s \left(\frac{\partial T}{\partial x}\omega n_x + \frac{\partial T}{\partial y}\omega n_y\right) ds - \int_A \left(\frac{\partial T}{\partial x}\frac{\partial \omega}{\partial x} + \frac{\partial T}{\partial y}\frac{\partial \omega}{\partial y}\right) dA = 0.$$

Simply put

$$\int_s \frac{\partial T}{\partial n}\omega ds - \int_A \left(\frac{\partial T}{\partial x}\frac{\partial \omega}{\partial x} + \frac{\partial T}{\partial y}\frac{\partial \omega}{\partial y}\right) dA = 0. \tag{5}$$

Considering the equalities:

$$\int_A \frac{\partial T}{\partial x}\frac{\partial \omega}{\partial x} dA = \int_s T\frac{\partial \omega}{\partial x} n_x ds - \int_A T\frac{\partial^2 \omega}{\partial x^2} dA \tag{6}$$

and

$$\int_A \frac{\partial T}{\partial y}\frac{\partial \omega}{\partial y} dA = \int_s T\frac{\partial \omega}{\partial y} n_y ds - \int_A T\frac{\partial^2 \omega}{\partial y^2} n_y dA. \tag{7}$$

Replacing equations (6) and (7) in the equation (5), we have:

$$\int_s \frac{\partial T}{\partial n}\omega ds - \int_s \left(T\frac{\partial \omega}{\partial x} n_x + T\frac{\partial \omega}{\partial y} n_y\right) ds + \int_A T\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) dA = 0.$$

$$\int_s \frac{\partial T}{\partial n}\omega ds - \int_s T\frac{\partial \omega}{\partial n} ds + \int_A T\Delta\omega dA = 0. \tag{8}$$

In order to obtain an integral equation that does not have domain integrals (area integrals) the function $\omega$ must be chosen so that the domain integral of Eq.(2.19) disappears. Any harmonic function, that is, a function that Laplacian is equal to zero, satisfies this requirement. However, for numerical reasons, the most suitable choice is the function whose Laplacian is the Dirac delta.

$$\Delta\omega = -\frac{\delta(x - x_d)}{k},$$

which implies that $\omega = T^*$. So you have:

$$\int_s \frac{\partial T}{\partial n} T^* ds - \int_s T\frac{\partial T^*}{\partial n} ds + \int_A T\frac{[-\delta(x - x_d)]}{k} dA, \tag{9}$$

where $x_d$ is the coordinate of the source point.

Taking the source point within the $A$ domain, by the property of the Dirac delta, we have:

$$\int_s \frac{\partial T}{\partial n} T^* ds - \int_s T\frac{\partial T^*}{\partial n} ds - \frac{T(x_d, y_d)}{k} = 0.$$

Multiplying the terms by $-k$, you get:

$$\int -k\frac{\partial T}{\partial n} T^* ds + \int_s T\left(\frac{k\partial T^*}{\partial n}\right) ds + T(x_d, y_d) = 0.$$

$$T(x_d, y_d) = \int_s T q^* ds - \int_s q T^* ds. \tag{10}$$

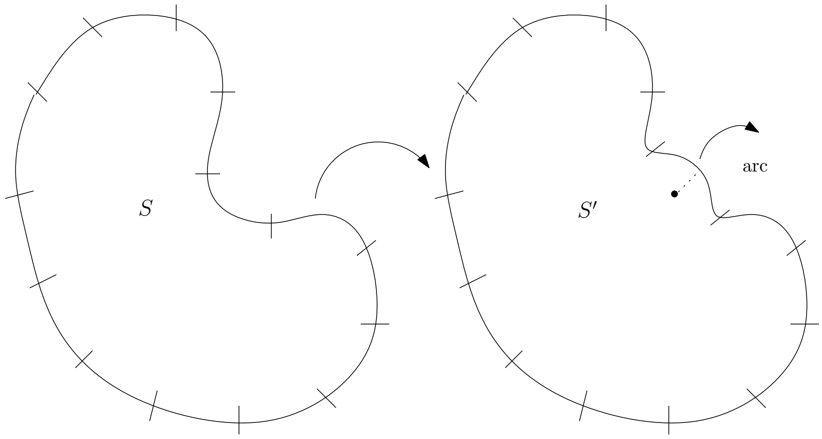


**Figure 1: Original and modified outlines.**

Eq.(10) is the *integral boundary equation* when the source point is inside the domain.

In order to consider the point $(x_d, y_d)$ on the boundary, a small modification is made to it, as shown in Fig. 1:

Thus, one has:

$$T(x_d, y_d) = \int_{s-s'} Tq^* ds - \int_{s-s'} T^* q ds + \int_{s^*} Tq^* ds - \int_{s^*} T^* q ds. \tag{11}$$

The flux fundamental solution is given by:

$$q^* = \frac{1}{2\pi r^2}[(x - x_d)n_x + (y - y_d)n_y],$$

with $r = \sqrt{(x - x_d)^2 + (y - y_d)^2}$,. Thus,

$$\int_{s^*} Tq^* ds = \int_{\theta_1}^{\theta_2} T \frac{1}{2\pi r^2}[(x - x_d)n_x + (y - y_d)n_y]\varepsilon d\theta.$$

In $s^*$, you have:

$$\vec{r} = (x - x_d)\vec{i} + (y - y_d)\vec{j},$$
$$|\vec{r}| = r = \sqrt{(x - x_d)^2 + (y - y_d)^2},$$
$$\vec{n} = \frac{(x - x_d)\vec{i} + (y - y_d)\vec{j}}{r},$$

where $\vec{n}$ is a unit vector. As $r_x = (x - x_d)$ and $r_y = (y - y_d)$, we have:

$$\vec{n} = \frac{r_x \vec{i} + r_y \vec{j}}{r}$$

with

$$n_x = \frac{r_x}{r} \quad \text{e} \quad n_y = \frac{r_y}{r}.$$

So

$$\int_{s^*}^{s} Tq^* ds = \int_{\theta_1}^{\theta_2} \frac{T}{2\pi r^2} \left( r_x \frac{r_x}{r} + r_y \frac{r_y}{r} \right) \varepsilon d\theta.$$

Noting that $r = \varepsilon$ for any $\theta$, we have;

$$\int_{s^*} Tq^* ds = \int_{\theta_1}^{\theta_2} \frac{T}{2\pi\varepsilon^2}\left(\frac{r_x^2 + r_y^2}{\varepsilon}\right)\varepsilon d\theta = \int_{\theta_1}^{\theta_2} \frac{T}{2\pi} d\theta.$$

By making $\varepsilon \longrightarrow 0$, $T$ takes the value of $T(d)$. Finally, there is

$$\int_{s^*} Tq^* ds = \frac{T(d)(\theta_2 - \theta_1)}{2\pi}.$$

The same analysis must be done for:

$$\int_{s^*} T^* q ds = \int_{\theta_1}^{\theta_2} \frac{-1}{2\pi k} \ln r q r d\theta.$$

As $r = \varepsilon$ = constant, we have:

$$\int_{s^*} T^* q ds = \frac{-1}{2\pi k}\varepsilon \ln \varepsilon \int_{\theta_1}^{\theta_2} q d\theta.$$

Making $\varepsilon \longrightarrow 0$, you get:

$$\int_{s^*} T^* q ds = \frac{-1}{2\pi k} \lim_{\varepsilon \to 0} \varepsilon \ln \varepsilon (\theta_2 - \ theta_1),$$

$$\int_{s^*} T^* q ds = 0.$$

Returning to the original equation, follows:

$$T(x_d, y_d) = \int_s Tq^* ds - \int_s T^* q ds + \frac{T(x_d, y_d)(\theta_2 - \theta_1)}{2\pi} - 0,$$

$$T(x_d, y_d)\left[1 - \frac{(\theta_2 - \theta_1)}{2\pi}\right] = \int_s Tq^* ds - \int_s T^* q ds,$$

$$T(x_d, y_d)\left[\frac{2\pi - (\theta_2 - \theta_1)}{2\pi}\right] = \int_s Tq^* ds - \int_s T^* q ds.$$

As shown in Fig. 2, $\theta_{int}$ is the inner angle of the boundary.

$$\frac{\theta_{int}}{2\pi} T(x_d, y_d) = \int_s Tq^* ds - \int_s T^* q ds,$$

which is the boundary integral when the source point belongs to the boundary.

When the source point does not belong to the boundary or the domain, due to the property of the Dirac delta, we have:

$$\int_s Tq^* ds - \int_s T^* q ds = 0. \tag{12}$$

Generally speaking, the integral boundary equation can be written as

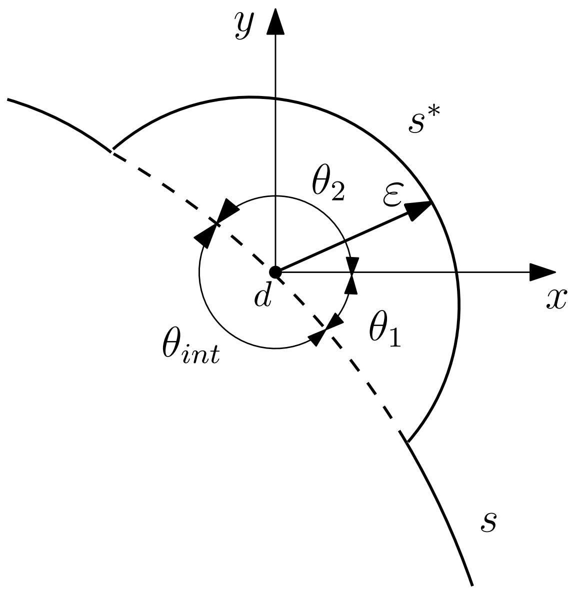


**Figure 2: Internal boundary angle.**

$$cT(x_d, y_d) = \int_s Tq^* ds - \int_s T^* q ds, \tag{13}$$

Where

$$c = \begin{cases} 1, & \text{if} \quad (x_d, y_d) \in \text{domain} \\ \frac{\theta_{int}}{2\pi}, & \text{if} \quad (x_d, y_d) \in \text{boundary} \\ 0, & \text{if} \quad (x_d, y_d) \notin \text{domain or boundary} \end{cases}$$

When the source point is at a smooth point of the boundary, that is, it is not a corner, you have:

$$c = \frac{\theta_{int}}{2\pi} = \frac{\pi}{2\pi} = \frac{1}{2}. \tag{14}$$

## 3 Integral equation for heat flux

To obtain an integral equation for the heat flux, it is necessary to derive the equation (10) in relation to the coordinates of the source point, that is:

$$\frac{\partial T(x_d, y_d)}{\partial x_d} = \frac{\partial}{\partial x_d}\left[\int_s Tq^* ds - \int_s qT^* ds\right]. \tag{15}$$

$$\frac{\partial T(x_d, y_d)}{\partial x_d} = \int_s T\frac{\partial q^*}{\partial x_d} ds - \int_s q\frac{\partial T^*}{\partial x_d} ds. \tag{16}$$

where:

$$\frac{\partial T^*}{\partial x_d} = \frac{r_x}{2\pi k r^2}, \tag{17}$$

$$\frac{\partial T^*}{\partial y_d} = \frac{r_y}{2\pi k r^2}, \tag{18}$$

$$\frac{\partial q^*}{\partial x_d} = \frac{\left[n_x\left(r_x^2 - r_y^2\right) + 2n_y r_x r_y\right]}{2\pi r^4} \tag{19}$$

and

$$\frac{\partial q^*}{\partial y_d} = \frac{\left[n_y\left(-r_x^2 + r_y^2\right) + 2n_x r_x r_y\right]}{2\pi r^4}. \tag{20}$$

## 4 Discretization of integral boundary equations

Basically, the MEC formulation transforms differential equations into integral boundary equations, thus eliminating domain discretization. These integrals can be solved numerically and analytically with the integration made along the boundary, which is discretized by dividing it into elements called *boundary elements* in which boundary conditions are prescribed.

Once the boundary integral is obtained, the next step is to discretize this equation so that the integrals along the boundary are written as the sum of integrals along parts of the boundary.

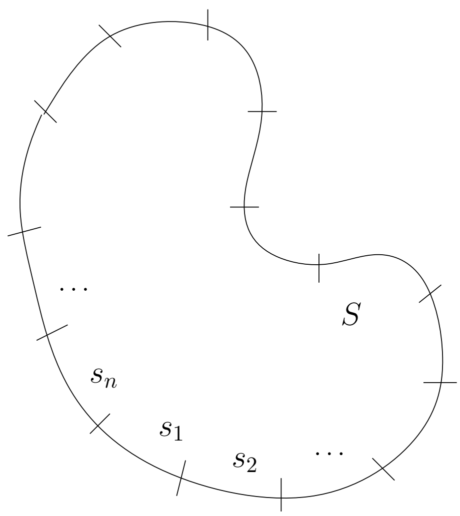


**Figure 3: Discretization of boundary in n parts.**

In this way, the integral boundary equation (13) is written as:

$$cT(x_d, y_d) = \sum_{j=1}^{n} \int_{s_j} Tq^* ds - \sum_{j=1}^{n} \int_{s_j} T^* q ds. \tag{21}$$

where

$$S = s_1 + s_2 + \ldots + s_n.$$

## 5 Constant boundary elements

In discretization using constant elements, the geometry is approximated by straight line segments with a node in the middle of each element. Thus, consider that the parts of the boundary $s_1, s_2, ..., s_n$ are approximated by straight segments and that along these segments, both temperature and flux are assumed as constants.

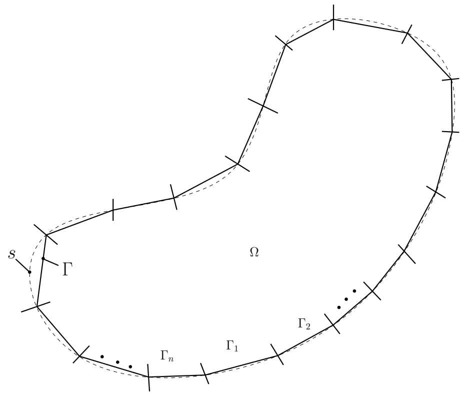


**Figure 4: boundary approximation by line segments.**

The $j$ node will always be at the center position of the $j$ element (it will always be in a smooth region of the boundary, so $c = \frac{1}{2}$). The integral equation is approximated by:

$$\frac{1}{2}T^{(i)}(x_d, y_d) = \sum_{j=1}^{n}\left[T_j \int_{\Gamma_j} q^* d\Gamma\right] - \sum_{j=1}^{n}\left[q_j \int_{\Gamma_j} T^* d\Gamma\right], \tag{22}$$

where $i$ corresponds to the node of the $i$-th element. Hence, we have:

$$-\frac{1}{2}T^{(i)}(x_d, y_d) + \sum_{j=1}^{n}\left[T_j \int_{\Gamma_j} q^* d\Gamma\right] = \sum_{j=1}^{n}\left[q_j \int_{\Gamma_j} T^* d\Gamma\right]. \tag{23}$$

Calling

$$H_{ij} = \begin{cases} \int_{\Gamma_j} q^* d\Gamma, & \text{se} \quad i \neq j \\ -\frac{1}{2} + \int_{\Gamma_j} q^* d\Gamma, & \text{se} \quad i = j \end{cases}$$

and

$$G_{ij} = \int_{\Gamma_j} T^* d\Gamma \tag{24}$$

you can write the matrix equation as follows:

$$\sum_{j=1}^{n}[H_{ij}T_j] = \sum_{j=1}^{n}[G_{ij}q_j]. \tag{25}$$

**Example 5.1** *In order to illustrate how to apply the boundary conditions and calculate the unknown variables, a unidirectional heat conduction problem with a discretization of one element per side will be analyzed (see Figure 5).*

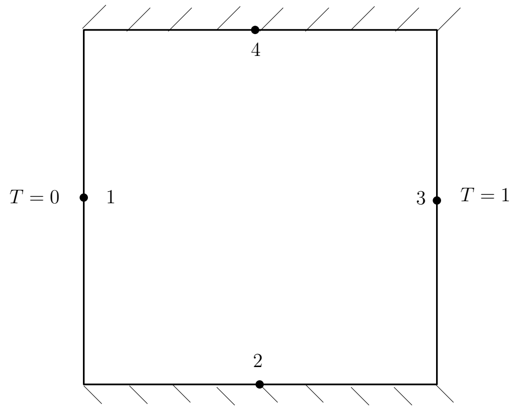


**Figure 5: Temperature and flux on plate.**

**Table 1: Qualification of variables in each node.**

| node | known variables | unknown variables |
|---|---|---|
| 1 | $T_1$ | $q_1$ |
| 2 | $q_2$ | $T_2$ |
| 3 | $T_3$ | $q_3$ |
| 4 | $q_4$ | $T_4$ |

*In this case, the known and unknown variables in the problem outline are given by Table (2.1).*

*Considering that the source point is at node 1 and subscribing the known variables with a slash, we have:*

$$H_{11}\bar{T}_1 + H_{12}T_2 + H_{13}\bar{T}_3 + H_{14}T_4 = G_{11}q_1 + G_{12}\bar{q}_2 + G_{13}q_3 + G_{14}\bar{q}_4,$$

*where $\bar{T}$ and $\bar{q}$ are known terms. Since there is only 1 equation and 4 unknown variables, three more equations must be generated. To do this, just place the source point on each of the nodes. For this reason the choice of the weight function $\omega$ must be that Lapaltian is equal to Dirac's delta and not Laplacian equal to zero. Thus, one has:*

*For the source point at node 2, we have:*

$$H_{21}\bar{T}_1 + H_{22}T_2 + H_{23}\bar{T}_3 + H_{24}\bar{T}_4 = G_{21}q_1 + G_{22}\bar{q}_2 + G_{23}q_3 + G_{24}\bar{q}_4.$$

*Likewise, the source point is made at nodes 3 and 4. The equations obtained can be written in matrix form, as:*

$$\begin{pmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{pmatrix} \begin{pmatrix} \bar{T}_1 \\ T_2 \\ \bar{T}_3 \\ T_4 \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{pmatrix} \begin{pmatrix} q_1 \\ \bar{q}_2 \\ q_3 \\ \bar{q}_4 \end{pmatrix}$$

*which can be briefly written as:*

$$[H]\{T\} = [G]\{q\}. \tag{26}$$

*Separating the known terms from the unknown, it follows:*

$$\begin{pmatrix} -G_{11} & H_{12} & -G_{13} & H_{14} \\ -G_{21} & H_{22} & -G_{23} & H_{24} \\ -G_{31} & H_{32} & -G_{33} & H_{34} \\ -G_{41} & H_{42} & -G_{43} & H_{44} \end{pmatrix} \begin{pmatrix} q_1 \\ T_2 \\ q_3 \\ T_4 \end{pmatrix} = \begin{pmatrix} -H_{11} & G_{12} & -H_{13} & G_{14} \\ -H_{21} & G_{22} & -H_{23} & G_{24} \\ -H_{31} & G_{32} & -H_{33} & G_{34} \\ -H_{41} & G_{42} & -H_{43} & G_{44} \end{pmatrix} \begin{pmatrix} \bar{T}_1 \\ \bar{q}_2 \\ \bar{T}_3 \\ \bar{q}_4 \end{pmatrix}$$

*So you can write*

$$[A]\{x\} = \{b\}, \tag{27}$$

*or yet*

$$\{x\} = [A]^{-1}\{b\}. \tag{28}$$

*Then, solve the linear system above and calculate the values of the unknown variables.*

### 5.1 Integration of matrices $[H]$ and $[G]$ when source point belongs to element

In this case, in the case of constant elements, the integration is done analytically, that is:

- Matrix H

$$H_{ij} = -\frac{1}{2} + \frac{1}{2\pi} \int_{\Gamma_j} \frac{r_x n_x + r_y n_y}{r} d\Gamma$$

Like

$$r_x n_x + r_y n_y = \vec{r}.\vec{n} = 0$$

$$\therefore H_{ij} = -\frac{1}{2} \tag{29}$$

- Matrix G

For constant boundary element, matrix G is given by:

$$G = -\frac{1}{2\pi k} \int_{\Gamma_j} \ln r d\Gamma \tag{30}$$

Thus, one has:

$$\begin{aligned} G_{ij} &= -\frac{1}{2\pi k} \times 2 \int_0^{\frac{L}{2}} \ln r dr \\ &= -\frac{1}{\pi k} \left(-r + r \ln r\right) \Big|_0^{\frac{L}{2}} \\ &= -\frac{1}{\pi k} \left( -\frac{L}{2} + \frac{L}{2} \ln \frac{L}{2} + 0 - \lim_{r \to 0} r \ln r \right). \\ &= \frac{L}{2\pi k} \left( 1 - \ln \frac{L}{2} \right) \end{aligned}$$

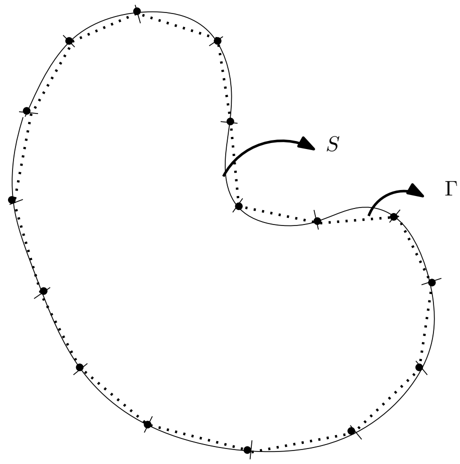


**Figure 6: Continuous linear elements.**

## 6 Continuous linear boundary elements

In linear element discretization, the geometry is approximated by a 1st degree polynomial, requiring two nodes in each element, one at each end of the element. Temperature and flux are also approximated by a 1st degree polynomial. The formulation is isoparametric, that is, the same shape functions used to interpolate the geometry are also used to interpolate the physical variables (temperature and flux).

In this case, the integral equation is given by:

$$cT(x_d, y_d) = \int_s Tq^* dS - \int_s T^* q dS.$$

Discretizing in continuous linear boundary elements, it follows:

$$cT(x_d, y_d) = \sum_{j=1}^{n_{elem}} \left[ \int_{\Gamma_j} Tq^* d\Gamma \right] - \sum_{j=1}^{n_{elem}} \left[ \int_{\Gamma_j} T^* q d\Gamma \right]$$

Observing that $T$ and $q$ are assumed with linear variation along the element, that is,

$$T = N_1 T_1 + N_2 T_2$$

and

$$q = N_1 q_1 + N_2 q_2.$$

where $T_1$ is the temperature at local node 1, $T_2$ the temperature at local node 2, $q_1$ is the flux at local node 1 and $q_2$ is the flux at local node 2, $N_1$ is the form function 1 and $N_2$ is the form function 2.

Likewise, it follows:

$$\begin{cases} x = N_1 x_1 + N_2 x_2 \\ y = N_1 y_1 + N_2 y_2 \end{cases}$$

Writing in matrix form, follows:

$$T = \left[ \begin{array}{cc} N_1 & N_2 \end{array} \right] \left[ \begin{array}{c} T_1 \\ T_2 \end{array} \right]$$

and

$$q = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

The discretized integral equation is then written as:

$$cT(x_d, y_d) = \sum_{j=1}^{n_{elem}} \left\{ \int_{\Gamma_j} \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}_j q^* d\Gamma \right\} - \sum_{j=1}^{n_{elem}} \left\{ \int_{\Gamma_j} T^* \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_j d\Gamma \right\}.$$

Since $T_1$, $T_2$, $q_1$ and $q_2$ are nodal values, it follows:

$$cT(x_d, y_d) = \sum_{j=1}^{n_{elem}} \left\{ \int_{\Gamma_j} \begin{bmatrix} N_1 & N_2 \end{bmatrix} q^* d\Gamma \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}_j \right\} - \sum_{j=1}^{n_{elem}} \left\{ \int_{\Gamma_j} \begin{bmatrix} N_1 & N_2 \end{bmatrix} T^* d\Gamma \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_j \right\}.$$

which can be written as follows:

$$cT(x_d, y_d) = \sum_{j=1}^{n_{elem}} \left\{ \begin{bmatrix} h_1 & h_2 \end{bmatrix}_j \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}_j - \begin{bmatrix} g_1 & g_2 \end{bmatrix}_j \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_j \right\},$$

Where

$$h_1 = \int_{\Gamma_j} N_1 q^* d\Gamma,$$

$$h_2 = \int_{\Gamma_j} N_2 q^* d\Gamma,$$

$$g_1 = \int_{\Gamma_j} N_1 T^* d\Gamma$$

and

$$g_2 = \int_{\Gamma_j} N_2 T^* d\Gamma.$$

**Example 6.1** *Applying the formulation developed in the heat conduction problem discussed above (Figure 7), the boundary conditions and unknown variables are given as shown in table 2. Note in Table 2 that the temperature is continuous at node $j$. In turn, the flux $q_j$ can be discontinuous, that is, the flux $q_j^a$, before the node $j$ can be different from the flux $q_j^d$, after the node $j$. However, given the order of the Laplace differential equation (second order), only one variable can be unknown per node.*

**Table 2: Qualification of the variables in each node for the given problem.**

| node | known variables | unknown variables |
|---|---|---|
| 1 | $T_1$ and $q_1^a$ | $q_1^d$ |
| 2 | $T_2$ and $q_2^d$ | $q_2^a$ |
| 3 | $T_3$ and $q_3^a$ | $q_3^d$ |
| 4 | $T_4$ and $q_4^d$ | $q_4^a$ |

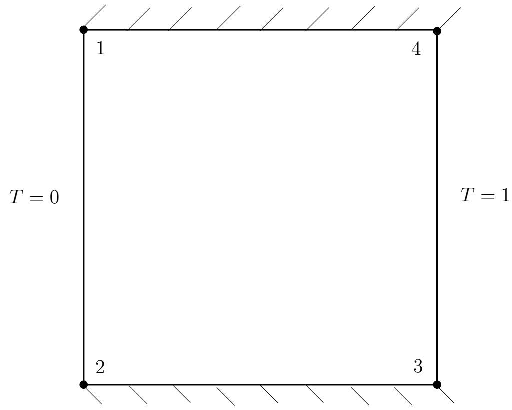


**Figure 7: Temperature and flux on plate.**

*Considering the source point at node 1, the integral equation is described as:*

$$\begin{aligned} cT_1 &= \begin{bmatrix} h_1 & h_2 \end{bmatrix}_1 \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}_1 + \begin{bmatrix} h_1 & h_2 \end{bmatrix}_2 \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}_2 + \cdots - \begin{bmatrix} g_1 & g_2 \end{bmatrix}_1 \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_1 - \\ & - \begin{bmatrix} g_1 & g_2 \end{bmatrix}_2 \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_2 \cdots . \end{aligned}$$

*Using the global node number, it follows:*

$$\begin{aligned} cT_1 &= \begin{bmatrix} h_1 & h_2 \end{bmatrix}_1 \begin{bmatrix} \bar{T}_1 \\ \bar{T}_2 \end{bmatrix} + \begin{bmatrix} h_1 & h_2 \end{bmatrix}_2 \begin{bmatrix} \bar{T}_2 \\ \bar{T}_3 \end{bmatrix} + \begin{bmatrix} h_1 & h_2 \end{bmatrix}_3 \begin{bmatrix} \bar{T}_3 \\ \bar{T}_4 \end{bmatrix} + \\ & + \begin{bmatrix} h_1 & h_2 \end{bmatrix}_4 \begin{bmatrix} \bar{T}_4 \\ \bar{T}_1 \end{bmatrix} - \begin{bmatrix} g_1 & g_2 \end{bmatrix}_1 \begin{bmatrix} q_1^d \\ q_2^a \end{bmatrix} - \begin{bmatrix} g_1 & g_2 \end{bmatrix}_2 \begin{bmatrix} \bar{q}_2^d \\ \bar{q}_3^a \end{bmatrix} - \\ & - \begin{bmatrix} g_1 & g_2 \end{bmatrix}_3 \begin{bmatrix} q_3^d \\ q_4^a \end{bmatrix} - \begin{bmatrix} g_1 & g_4 \end{bmatrix}_1 \begin{bmatrix} \bar{q}_4^d \\ \bar{q}_1^a \end{bmatrix}, \end{aligned}$$

*Writing the global G and H matrices, follows:*

$$\begin{aligned} & H_{11}T_1 + H_{12}T_2 + H_{13}T_3 + H_{14}T_4 = G_{11}^d q_1^d + G_{12}^a q_2^a \\ & + G_{12}^d \bar{q}_2^d + G_{13}^a \bar{q}_3^a + G_{13}^d q_3^d + G_{14}^a q_4^a + G_{14}^d \bar{q}_4^d + G_{11}^a \bar{q}_1^a. \end{aligned} \tag{31}$$

*Note that there is 1 equation and 4 unknown variables. In order to generate 3 more equations, just place the source point at the other 3 nodes. Hence, the following matrix equation is obtained:*

$$
\begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix} \begin{bmatrix} \bar{T}_1 \\ \bar{T}_2 \\ \bar{T}_3 \\ \bar{T}_4 \end{bmatrix} =
$$

$$
\begin{bmatrix} G^d_{11} & G^a_{12} & G^d_{12} & G^a_{13} & G^d_{13} & G^a_{14} & G^d_{14} & G^a_{11} \\ G^d_{21} & G^a_{22} & G^d_{22} & G^a_{23} & G^d_{23} & G^a_{24} & G^d_{24} & G^a_{21} \\ G^d_{31} & G^a_{32} & G^d_{32} & G^a_{33} & G^d_{33} & G^a_{34} & G^d_{34} & G^a_{31} \\ G^d_{41} & G^a_{42} & G^d_{42} & G^a_{43} & G^d_{43} & G^a_{44} & G^d_{44} & G^a_{41} \end{bmatrix} \begin{bmatrix} q^d_1 \\ q^a_2 \\ \bar{q}^d_2 \\ \bar{q}^a_3 \\ q^d_3 \\ \bar{q}^a_4 \\ q^d_4 \\ \bar{q}^a_1 \end{bmatrix}. \tag{32}
$$

*Manipulating the matrix equation so that the unknown terms are all on the left side and the other terms on the right side, follows:*

$$
\begin{bmatrix} -G^d_{11} & -G^a_{12} & -G^d_{13} & -G^a_{14} \\ -G^d_{21} & -G^a_{22} & -G^d_{23} & -G^a_{24} \\ -G^d_{31} & -G^a_{32} & -G^d_{33} & -G^a_{34} \\ -G^d_{41} & -G^a_{42} & -G^d_{43} & -G^a_{44} \end{bmatrix} \begin{bmatrix} q^d_1 \\ q^a_2 \\ q^d_3 \\ \bar{q}^a_4 \end{bmatrix}
$$

$$
= \begin{bmatrix} -H_{11} & -H_{12} & G^d_{12} & G^a_{13} & -H_{13} & -H_{14} & G^d_{14} & G^a_{11} \\ -H_{21} & -H_{22} & G^d_{22} & G^a_{23} & -H_{23} & -H_{24} & G^d_{24} & G^a_{21} \\ -H_{11} & -H_{32} & G^d_{32} & G^a_{33} & -H_{33} & -H_{34} & G^d_{34} & G^a_{31} \\ -H_{11} & -H_{42} & G^d_{42} & G^a_{43} & -H_{43} & -H_{44} & G^d_{44} & G^a_{41} \end{bmatrix} \begin{bmatrix} \bar{T}_1 \\ \bar{T}_2 \\ q^d_2 \\ q^a_3 \\ \bar{T}_3 \\ \bar{T}_4 \\ \bar{q}^d_4 \\ \bar{q}^a_1 \end{bmatrix}, \tag{33}
$$

*that can be written in linear form as:*

$$
[A]\{x\} = \{b\} \tag{34}
$$

**Example 6.2** *Applying the formulation developed in the heat conduction problem represented in Figure 8, the boundary conditions and unknown variables are given as shown in table 3. Note in Table 2 that the temperature is continuous at node $j$. In turn, the flux $q_j$ can be discontinuous, that is, the flux $q^a_j$, before the node $j$ can be different from the flux $q^d_j$, after the node $j$. However, given the order of the Laplace differential equation (second order), only one variable can be unknown per node.*

**Table 3: Qualification of the variables in each node for the given problem.**

| node | known variables | unknown variables |
|---|---|---|
| 1 | $q_1^a$ and $q_1^d$ | $T_1$ |
| 2 | $T_2$ and $q_2^a$ | $q_2^d$ |
| 3 | $T_3$ | $q_3^a = q_3^d = q_3$ |
| 4 | $T_4$ and $q_4^d$ | $q_4^a$ |

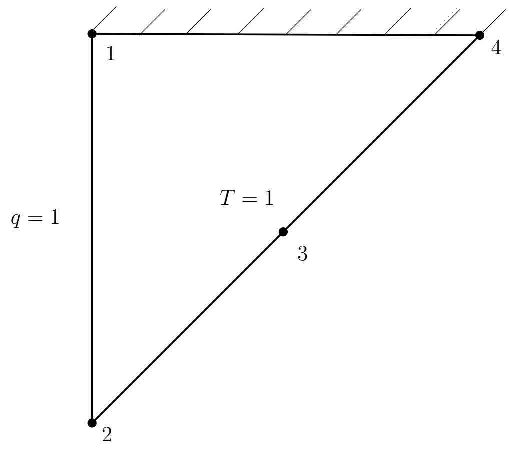


**Figure 8: Temperature and flux on plate.**

$$
\begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix} \begin{bmatrix} T_1 \\ \bar{T}_2 \\ \bar{T}_3 \\ \bar{T}_4 \end{bmatrix}
$$
$$
= \begin{bmatrix} G_{11}^d & G_{12}^a & G_{12}^d & G_{13}^a & G_{13}^d & G_{14}^a & G_{14}^d & G_{11}^a \\ G_{21}^d & G_{22}^a & G_{22}^d & G_{23}^a & G_{23}^d & G_{24}^a & G_{24}^d & G_{21}^a \\ G_{31}^d & G_{32}^a & G_{32}^d & G_{33}^a & G_{33}^d & G_{34}^a & G_{34}^d & G_{31}^a \\ G_{41}^d & G_{42}^a & G_{42}^d & G_{43}^a & G_{43}^d & G_{44}^a & G_{44}^d & G_{41}^a \end{bmatrix} \begin{bmatrix} \bar{q}_1^d \\ \bar{q}_2^a \\ q_2^d \\ q_3^a = q_3 \\ q_3^d = q_3 \\ q_4^a \\ \bar{q}_4^d \\ \bar{q}_1^a \end{bmatrix}. \tag{35}
$$

*Manipulating the matrix equation so that the unknown terms are all on the left side and the other terms on the right side, follows:*

$$\left[\begin{array}{cccc} H_{11} & -G_{12}^d & (-G_{13}^a - G_{13}^d) & -G_{14} \\ H_{21} & -G_{22}^d & (-G_{23}^a - G_{23}^d) & -G_{24} \\ H_{31} & -G_{32}^d & (-G_{33}^a - G_{33}^d) & -G_{34} \\ H_{41} & -G_{42}^d & (-G_{43}^a - G_{43}^d) & -G_{44} \end{array}\right]\left[\begin{array}{c} T_1 \\ q_2^d \\ q_3 \\ q_4^a \end{array}\right]$$

$$= \left[\begin{array}{cccccccc} G_{11}^d & G_{12}^a & -H_{12} & -H_{13} & 0 & -H_{14} & G_{14}^d & G_{11}^a \\ G_{21}^d & G_{22}^a & -H_{22} & -H_{23} & 0 & -H_{24} & G_{24}^d & G_{21}^a \\ G_{31}^d & G_{32}^a & -H_{32} & -H_{33} & 0 & -H_{34} & G_{34}^d & G_{31}^a \\ G_{41}^d & G_{42}^a & -H_{42} & -H_{43} & 0 & -H_{44} & G_{44}^d & G_{41}^a \end{array}\right]\left[\begin{array}{c} \bar{q}_1^d \\ \bar{q}_2^a \\ \bar{T}_2 \\ \bar{T}_3 \\ 0 \\ \bar{T}_4 \\ \bar{q}_4^d \\ \bar{q}_1^a \end{array}\right], \quad (36)$$

*that can be written in linear form as:*

$$[A]\{x\} = \{b\} \quad (37)$$

### 6.1 Algorithm to apply boundary conditions

As seen in the examples 6.1 and 6.2, the procedures to apply the boundary conditions when you have continuous elements (where a node is shared by 2 elements) are more complex than for discontinuous elements. This section presents a simple algorithm that does this task well. Although the case shown is restricted to continuous linear elements, this algorithm can be easily extended to other types of continuous elements, both in the formulations of 2D boundary elements and 3D boundary elements.

Initially, assume that an array $[T_{pr}]$ will be created that contains information about the nodes at which the temperature is prescribed (known). This matrix has 5 columns and the number of rows is equal to the number of nodes for which the temperature is known. For ease of understanding, consider that the matrix columns are represented by five vectors $\{a_1\}$, $\{a_2\}$, $\{a_3\}$, $\{a_4\}$ and $\{a_5\}$. Thus, the line $i$ of the matrix $[T_{pr}]$ is given by:

$$T_{pr_i} = \left[\begin{array}{ccccc} a_{1i} & a_{2i} & a_{3i} & a_{4i} & a_{5i} \end{array}\right] \quad (38)$$

where each element $i$ of the vectors $\{a_1\}$, $\{a_2\}$, $\{a_3\}$, $\{a_4\}$ and $\{a_5\}$ contains:

$a_{1i}$: number of the $i-$th node with known temperature.
$a_{2i}$: number of the first element **with prescribed temperature** to which this node belongs.
$a_{3i}$: local node number in this element.
$a_{4i}$: if the temperature is also prescribed in the second element to which this node belongs, then $a_{4i}$ will contain the number of this element, otherwise it will contain zero.
$a_{5i}$: if $a_{4i}$ is non-zero, $a_{5i}$ will contain the local number of the node in the second element, otherwise it will contain zero.

In the definition of $a_{2i}$, the term ”with prescribed temperature” is in bold to draw attention to the fact that, if the temperature is prescribed in only one of the elements to which the node $i$ belongs , the element that does not have a prescribed temperature should not be considered in the matrix $[T_{pr}]$.

The matrices $[T_{pr}]$ for the examples 6.1 and 6.2 are given, respectively, by:

$$[T_{pr}] = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 \\ 4 & 3 & 2 & 0 & 0 \end{bmatrix} \tag{39}$$

and

$$[T_{pr}] = \begin{bmatrix} 2 & 2 & 1 & 0 & 0 \\ 3 & 2 & 2 & 3 & 1 \\ 4 & 3 & 2 & 0 & 0 \end{bmatrix} \tag{40}$$

Once the matrix $[T_{pr}]$ has been constructed, the exchange of the columns of the matrices $[H]$ and $[G]$ follows the following algorithm:

For i = 1 up to the number of nodes with known temperature:

- $i_{no} = T_{pr\,i1}$; (number of the $i-$th node with known temperature);
- $i_{el} = T_{pr\,i2}$; (first element with prescribed temperature that contains this node);
- $i_{noloc} = T_{pr\,i3}$; (local node number in this element);
- $ind_H = i_{no}$; (index of the $[H]$ matrix column that will be swapped);
- $ind_G = 2 \times i_{el} + i_{noloc} - 2$; (index of the $[G]$ matrix column that will be swapped);
- the vector $\{exchange\}$ receives the column $ind_G$ of the matrix $[G]$;
- column $ind_G$ of matrix $[G]$ receives column $ind_H$ of matrix $[H]$ with inverted sign;
- column $ind_H$ of matrix $[H]$ receives vector $\{exchange\}$ with inverted sign;
- If $T_{pr\,i4}$ is non-zero $\Rightarrow$ the temperature is also known in the second element to which the node $i_{no}$ belongs:
  - $i_{el} = T_{pr\,i4}$; (number of the second element to which the node belongs);
  - $i_{noloc} = T_{pr\,i5}$; (local number of this node in the second element);
  - $ind_G = 2 \times i_{el} + i_{noloc} - 2$; (index of the column of the matrix $[G]$ which will be assigned zero);
  - Subtracts from the elements of the column $ind_H$ of the matrix $[H]$ the value of the elements of the column $ind_G$ of the matrix $[G]$;
  - Assign zeros in column $ind_G$ of matrix $[G]$;
- End of If;

End of For.

### 6.2 Integration of matrices $[H]$ and $[G]$ when source point does not belong to element

The integration of the terms of the arrays $[H]$ and $[G]$ when the source point does not belong to the element is regular and does not present great differences in relation to the integration of the constant element. To avoid unnecessary repetition, the integration for the linear element will not be detailed.

### 6.3 Matrix integration $[G]$ when source point belongs to element

The integration of the matrix $[G]$ when the source point belongs to the element is done analytically, in the same way as the constant element.

As already seen, the element's geometry is approximated by:

$$\begin{aligned} x = N_1 x_1 + N_2 x_2 &= \frac{1}{2}(1-\xi)x_1 + \frac{1}{2}(1+\xi)x_2 \\ &= \frac{x_1 - \xi x_1 + x_2 + \xi x_2}{2} = \frac{1}{2}\left[(x_2 - x_1)\,\xi + x_2 + x_1\right] \end{aligned} \tag{41}$$

and

$$y = \frac{1}{2}\left[(y_2 - y_1)\,\xi + y_2 + y_1\right]. \tag{42}$$

The coordinate $x$ of the source point is given by $x_d = x(\xi = \xi_d)$ and $y_d = y(\xi = \xi_d)$, where $\xi_d = -1$ for the source point at node 1 and $\xi_d = +1$ for the source point at node 2. Hence:

$$x_d = \frac{1}{2}\left[(x_2 - x_1)\,\xi_d + x_2 + x_1\right], \tag{43}$$

$$y_d = \frac{1}{2}\left[(y_2 - y_1)\,\xi_d + y_2 + y_1\right] \tag{44}$$

and

$$r = \sqrt{(x - x_d)^2 + (y - y_d)^2} = \sqrt{r_x^2 + r_y^2}, \tag{45}$$

where

$$r_x = x - x_d = \frac{1}{2}\left[(x_2 - x_1)\xi + x_2 + x_1\right] - \left\{\frac{1}{2}\left[(x_2 - x_1)\xi_d + x_2 + x_1\right]\right\} \tag{46}$$

$$r_x = \frac{1}{2}(x_2 - x_1)(\xi - \xi_d) \tag{47}$$

Likewise, you have:

$$r_y = \frac{1}{2}(y_2 - y_1)(\xi - \xi_d) \tag{48}$$

and

$$r = \sqrt{\left[\frac{1}{2}(x_2 - x_1)(\xi - \xi_d)\right]^2 + \left[\frac{1}{2}(y_2 - y_1)(\xi - \xi_d)\right]^2}$$
$$= \frac{1}{2}(\xi - \xi_d)\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \frac{1}{2}(\xi - \xi_d)L \quad (49)$$

The terms of the matrix $[G]$ are given by:

$$g_1 = \int_{\Gamma_j} T^* N_1 d\Gamma \quad (50)$$

and

$$g_2 = \int_{\Gamma_j} T^* N_2 d\Gamma. \quad (51)$$

In this way, you have:

$$g_1 = \int_{\Gamma_j} T^* N_1 d\Gamma = \int_{-1}^{1} T^* N_1 \frac{d\Gamma}{d\xi} d\xi$$
$$= \int_{-1}^{1} \frac{-1}{2\pi k} \log(r) \frac{L}{2}\frac{1}{2}(1 - \xi) d\xi$$
$$= \frac{-L}{8\pi k} \int_{-1}^{1} \log\left[L\left(\frac{\xi - \xi_d}{2}\right)\right](1 - \xi) d\xi \quad (52)$$

- Source point at node 1: $\xi_d = -1$.

$$g_1 = \frac{-L}{8\pi k}\left[\int_{-1}^{1} \log\left(\frac{\xi + 1}{2}\right)(1 - \xi) d\xi + \int_{-1}^{1} \log(L)(1 - \xi) d\xi\right] \quad (53)$$

Making

$$\eta = \frac{\xi + 1}{2} \Rightarrow \frac{d\eta}{d\xi} = \frac{1}{2} \quad (54)$$

one has:

$$\eta(\xi = -1) = \frac{-1 + 1}{2} = 0 \quad (55)$$

$$\eta(\xi = 1) = \frac{1 + 1}{2} = 1 \quad (56)$$

$$\xi = 2\eta - 1 \Rightarrow 1 - \xi = 1 - 2\eta + 1 = 2(1 - \eta) \quad (57)$$

Hence, we have:

$$g_1 = -\frac{L}{8\pi k}\left[\int_0^1 \log(\eta)2(1-\eta)\frac{d\xi}{d\eta}d\eta + \ log(L)\left(\xi - \frac{\xi^2}{2}\right)\Bigg|_{-1}^{1}\right]$$
$$= -\frac{L}{8\pi k}\left\{\int_0^1 2(1-\eta)2\log(\eta)d\eta\right.$$
$$\left. + \log(L)\left[1-\left(\frac{1}{2}\right)^2 - (-1) + \left(\frac{-1}{2}\right)^2\right]\right\}$$
$$= -\frac{L}{8\pi k}\left[4\left(\int_0^1 \log(\eta)d\eta - \int_0^1 \eta\log(\eta)d\eta\right) + 2\log(L)\right] \quad (58)$$

$$g_1 = \frac{L}{4\pi k}\left[\frac{3}{2} - \log(L)\right]; \quad (59)$$

The integral $g_2$ is not singular when the source point is node 1 because $N_2 = 0$ at node 1, where $T^* \to \infty$.

- Source point at node 2: $\xi_d = 1$.

The integral $g_1$ is not singular when the source point is node 2 because $N_1 = 0$ at node 2, where $T^* \to \infty$.

$$\int_{-1}^{1} N_1 T^* \frac{d\Gamma}{d\xi}d\xi\Bigg|_{\xi_d=-1} = \int_{-1}^{1} N_2 T^* \frac{d\Gamma}{d\xi}d\xi\Bigg|_{\xi_d=1} \quad (60)$$

This way you have:

$$g_2 = \frac{L}{4\pi k}\left[\frac{3}{2} - \log(L)\right]. \quad (61)$$

### 6.4 Indirect method for calculating the diagonal of the matrix $[H]$

The singular terms of the matrix $[H]$ can also be calculated analytically, just as it was done for constant elements. However, since the nodes are now at the ends of the element rather than at the center, the source point may not belong to a smooth boundary if it is a corner node. Then, you must calculate the internal angle $\theta_{int}$ because the term $c$ of the equation (13) is no longer equal to 1/2. Although this calculation does not present great difficulties, there is an alternative implementation that is usually preferred when dealing with continuous elements. This implementation does not make the integration explicitly but uses a property of the matrix $[H]$ resulting from the modeling of a body under constant temperature. Without losing the generality, consider that all nodes of a body meet the temperature $T = 1$. In this case, the flux will be null on all nodes, that is, $q = 0$ on all nodes. In this way, the matrix equation is rewritten as:

$$[H]\{1\} = [G]\{0\} \quad (62)$$

where $\{1\}$ is a vector with all elements equal to 1 and $\{0\}$ is a vector with all elements equal to zero. In this case, it is easy to see that:

$$\sum_{j=1}^{N} H_{ij} = 0, \text{ for } i = 1, 2, ..., N. \tag{63}$$

where $N$ is the number of nodes.

Hence, the diagonal terms of the matrix $[H]$ can be calculated as follows:

$$H_{ii} = \sum_{j=1}^{N} H_{ij}, \text{ with } i \neq j, \text{ for } i = 1, 2, ..., N, \tag{64}$$

since all terms outside the diagonal are regular integrals and have been previously computed.

## 7 Continuous quadratic boundary elements

In discretization using quadratic elements, the geometry is approximated by a quadratic function along each element, requiring three nodal points per element as shown in Fig. 9.

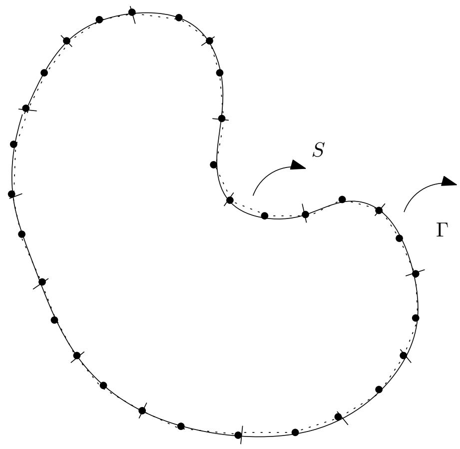


**Figure 9: Continuous quadratic elements.**

Thus temperature and flux are approximated as follows:

$$\begin{aligned} T &= N_1 T_1 + N_2 T_2 + N_3 T_3 \\ q &= N_1 q_1 + N_2 q_2 + N_3 q_3 \end{aligned}$$

where $T_1$ is the temperature at local node 1, $T_2$ the temperature at local node 2, $T_3$ the temperature at local node 3, $q_1$ is the flux at local node 1, $q_2$ is the flux at local node 2, $q_3$ is flux at local node 3, $N_1$ is form function 1, $N_2$ is form function 2, and $N_3$ is form function 3.

The continuous quadratic form functions $N_1$, $N_2$ and $N_3$ are given by:

$$N_1 = \frac{\xi}{2}(\xi - 1) \tag{65}$$

$$N_2 = (1-\xi)(1+\xi) = 1-\xi^2 \tag{66}$$

$$N_3 = \frac{\xi}{2}(\xi+1) \tag{67}$$

In this case, the integral equation is given by:

$$cT(d) = \int_s Tq^* dS - \int_s T^* q dS.$$

Discretizing in continuous quadratic boundary elements, it follows:

$$cT(d) = \sum_{j=1}^{n_{elem}} \left[ \int_{\Gamma_j} Tq^* d\Gamma \right] - \sum_{j=1}^{n_{elem}} \left[ \int_{\Gamma_j} T^* q d\Gamma \right]$$

Likewise, it follows:

$$\begin{cases} x = N_1 x_1 + N_2 x_2 + N_3 x_3 \\ y = N_1 y_1 + N_2 y_2 + N_3 y_3 \end{cases}$$

Writing in matrix form, follows:

$$T = \left[ \begin{array}{ccc} N_1 & N_2 & N_3 \end{array} \right] \left[ \begin{array}{c} T_1 \\ T_2 \\ T_3 \end{array} \right]$$

and

$$q = \left[ \begin{array}{ccc} N_1 & N_2 & N_3 \end{array} \right] \left[ \begin{array}{c} q_1 \\ q_2 \\ q_3 \end{array} \right].$$

The discretized integral equation is then written as:

$$\begin{aligned} cT(d) = & \sum_{j=1}^{n_{elem}} \left\{ \int_{\Gamma_j} \left[ \begin{array}{ccc} N_1 & N_2 & N_3 \end{array} \right] \left[ \begin{array}{c} T_1 \\ T_2 \\ T_3 \end{array} \right]_j q^* d\Gamma \right\} \\ & - \sum_{j=1}^{n_{elem}} \left\{ \int_{\Gamma_j} T^* \left[ \begin{array}{ccc} N_1 & N_2 & T_3 \end{array} \right] \left[ \begin{array}{c} q_1 \\ q_2 \\ q_3 \end{array} \right]_j d\Gamma \right\}. \end{aligned} \tag{68}$$

Since $T_1$, $T_2$, $T_3$, $q_1$, $q_2$ and $q_3$ are nodal values, it follows:

$$\begin{aligned} cT(d) = & \sum_{j=1}^{n_{elem}} \left\{ \int_{\Gamma_j} \left[ \begin{array}{ccc} N_1 & N_2 & N_3 \end{array} \right] q^* d\Gamma \left[ \begin{array}{c} T_1 \\ T_2 \\ T_3 \end{array} \right]_j \right\} \\ & - sum_{j=1}^{n_{elem}} \left\{ \int_{\Gamma_j} \left[ \begin{array}{ccc} N_1 & N_2 & N_3 \end{array} \right] T^* d\Gamma \left[ \begin{array}{c} q_1 \\ q_2 \\ q_3 \end{array} \right]_j \right\}, \end{aligned} \tag{69}$$

which can be written as follows:

$$cT(d) = \sum_{j=1}^{n_{elem}} \left\{ \begin{bmatrix} h_1 & h_2 & h_3 \end{bmatrix}_j \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}_j - \begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}_j \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}_j \right\},$$

where

$$h_1 = \int_{\Gamma_j} N_1 q^* d\Gamma,$$

$$h_2 = \int_{\Gamma_j} N_2 q^* d\Gamma,$$

$$h_3 = \int_{\Gamma_j} N_3 q^* d\Gamma,$$

$$g_1 = \int_{\Gamma_j} N_1 T^* d\Gamma,$$

$$g_2 = \int_{\Gamma_j} N_2 T^* d\Gamma$$

and

$$g_3 = \int_{\Gamma_j} N_3 T^* d\Gamma.$$

### 7.1 Integration of matrices $[H]$ and $[G]$ when source point does not belong to element

The integration of the terms of the arrays $[H]$ and $[G]$ when the source point does not belong to the element is regular and does not present great differences in relation to the integration of the constant element. To avoid unnecessary repetition, the integration for the quadratic element will not be detailed.

### 7.2 Matrix integration $[H]$ and $[G]$ when source point belongs to element

As already shown, the MEC presents some integrals of singular functions (functions that tend to infinity). In the case of the developed formulation, singular integrals are of two types:

1. In the matrix $[G]$ it is of the form $\log r$ which is called a weak singularity (improper integral);
2. In the matrix $[H]$ it is of the form $\frac{1}{r}$ which is called strong singularity (integral in the sense of Cauchy's principal value);

Therefore, the treatment of strong singularity can be done indirectly due to the properties of the matrix $[H]$, as shown in section 6.4. In the case of the matrix $[G]$, there are two possibilities, either numerically or analytically, the latter being only recommended for constant or linear elements. In the case of higher order form functions (quadratic, for example), the Jacobian of the transformation from $\Gamma$ to $\xi$ is no longer constant along the element, making the analytical treatment unfeasible. Therefore, numerical treatment is recommended.

Singular integrals of the order $(\log r)$ can be efficiently evaluated by the Gauss quadrature with a cubic variable transformation, as proposed by Telles Telles [1987], which exactly cancels the logarithmic singularity. Another possibility is the use of the logarithmic Gauss quadrature Brebbia and Dominguez [1992] which is among the most used numerical methods for the treatment of integrals with weak singularity in two-dimensional problems $(\log r)$.

The integration of the terms of the matrix $[H]$ for continuous quadratic elements is done indirectly, as already described in the section 6.4 for continuous linear elements.

The integration of the terms of the matrix $[G]$ for continuous quadratic elements is done using the logarithmic quadrature of Gauss, as will be detailed in the following paragraphs.

The coordinate $x$ of a point belonging to a quadratic element is approximated by:

$$\begin{aligned} x = N_1 x_1 + N_2 x_2 + N_3 x_3 &= \frac{\xi}{2}(\xi - 1)x_1 + (1-\xi^2)x_2 + \frac{\xi}{2}(\xi+1)x_3 \\ &= \frac{1}{2}\xi^2\left(x_1 - 2x_2 + x_3\right) + \frac{1}{2}\xi\left(x_3 - x_1\right)) + x_2 \end{aligned} \tag{70}$$

Likewise, you have:

$$y = \frac{1}{2}\xi^2\left(y_1 - 2y_2 + y_3\right) + \frac{1}{2}\xi\left(y_3 - y_1\right) + y_2 \tag{71}$$

The source point has coordinate $(x_d, y_d)$, where $x_d = x(\xi = \xi_d)$ and $y_d = y(\xi = \xi_d)$. Thus, we have $\xi_d = -1$ for the source point at node 1, $\xi_d = 0$ for the source point at node 2 and $\xi_d = +1$ for the source point at node 3. Hence, we have:

$$x_d = \frac{1}{2}\xi_d^2\left(x_1 - 2x_2 + x_3\right) + \frac{1}{2}\xi_d\left(x_3 - x_1\right) + x_2 \tag{72}$$

$$y_d = \frac{1}{2}\xi_d^2\left(y_1 - 2y_2 + y_3\right) + \frac{1}{2}\xi_d\left(y_3 - y_1\right) + y_2 \tag{73}$$

$$r = \sqrt{(x - x_d)^2 + (y - y_d)^2} = \sqrt{r_x^2 + r_y^2} \tag{74}$$

where

$$r_x = x - x_d = \frac{1}{2}\left(\xi^2 - \xi_d^2\right)\left(x_1 - 2x_2 + x_3\right) + \frac{1}{2}\left(\xi - \xi_d\right)\left(x_3 - x_1\right) \tag{75}$$

$$r_x = \frac{1}{2}\left(\xi - \xi_d\right)\left[\left(x_1 - 2x_2 + x_3\right)\left(\xi + \xi_d\right) + x_3 - x_1\right] \tag{76}$$

Likewise, you have:

$$r_y = \frac{1}{2}\left(\xi - \xi_d\right)\left[\left(y_1 - 2y_2 + y_3\right)\left(\xi + \xi_d\right) + y_3 - y_1\right] \tag{77}$$

and

$$r = \frac{1}{2}(\xi - \xi_d)\left\{\left[(x_1 - 2x_2 + x_3)(\xi + \xi_d) + x_3 - x_1\right]^2 + \left[(y_1 - 2y_2 + y_3)(\xi + \xi_d) + y_3 - y_1\right]^2\right\}^{\frac{1}{2}} \tag{78}$$

Calling

$$r_A = \frac{1}{2}(\xi - \xi_d) \tag{79}$$

and

$$r_B = \left\{\left[(x_1 - 2x_2 + x_3)(\xi + \xi_d) + x_3 - x_1\right]^2 + \left[(y_1 - 2y_2 + y_3)(\xi + \xi_d) + y_3 - y_1\right]^2\right\}^{\frac{1}{2}} \tag{80}$$

one has to

$$r = r_A r_B \tag{81}$$

where $r_B > 0$.

$$[g] = \int T^* \left[N_1 \quad N_2 \quad N_3\right] d\Gamma = \left[g_1 \quad g_2 \quad g_3\right] \tag{82}$$

where

$$g_1 = \int_{\Gamma_j} T^* N_1 d\Gamma, \tag{83}$$

$$g_2 = \int_{\Gamma_j} T^* N_2 d\Gamma \tag{84}$$

and

$$g_3 = \int_{\Gamma_j} T^* N_3 d\Gamma. \tag{85}$$

The integral $g_1$ is given by:

$$\begin{aligned} g_1 &= \int_{\Gamma_j} T^* N_1 d\Gamma = \int_{-1}^{1} T^* N_1 \frac{d\Gamma}{d\xi} d\xi \\ &= \int_{-1}^{1} \frac{-1}{2\pi k} \log(r_A r_B) N_1 \frac{d\Gamma}{d\,xi} d\xi \\ &= \frac{-1}{2\pi k} \int_{-1}^{1} \left[\log(r_A) + \log(r_B)\right] N_1 \frac{d\Gamma}{d\,xi} d\xi = g_{1s} + g_{1ns} \end{aligned} \tag{86}$$

where

$$g_{1s} = \frac{-1}{2\pi k} \int_{-1}^{1} \log(r_A) N_1 \frac{d\Gamma}{d\xi} d\xi \tag{87}$$

is a weak singularity integral that will be integrated using logarithmic Gauss quadrature and

$$g_{1ns} = \frac{-1}{2\pi k} \int_{-1}^{1} \log(r_B) N_1 \frac{d\Gamma}{d\xi} d\xi \tag{88}$$

is a regular (non-singular) integral that will be integrated using standard Gaussian quadrature.

- Source point at node 1: $\xi_d = -1$.

$$g_{1s} = \frac{-1}{2\pi k} \int_{-1}^{1} \log(r_A) \frac{\xi}{2}(\xi - 1) \frac{d\Gamma}{d\xi} d\xi \tag{89}$$

Making

$$\eta = \frac{\xi + 1}{2} \Rightarrow \frac{d\eta}{d\xi} = \frac{1}{2} \tag{90}$$

one has:

$$\eta(\xi = -1) = \frac{-1 + 1}{2} = 0 \tag{91}$$

$$\eta(\xi = 1) = \frac{1 + 1}{2} = 1 \tag{92}$$

$$\xi = 2\eta - 1 \Rightarrow 1 - \xi = 1 - 2\eta + 1 = 2(1 - \eta) \tag{93}$$

$$r_A = \frac{\xi - \xi_d}{2} = \eta \tag{94}$$

Hence, we have:

$$g_{1s} = \frac{-1}{2\pi k} \int_{0}^{1} \log(\eta) \, N_1(\xi(\eta)) \frac{d \; Gamma}{d\xi} \frac{d\xi}{d\eta} d\eta = \frac{-1}{\pi k} \int_{0}^{1} \log(\eta) \, N_1(\xi(\eta)) \frac{d\Gamma}{d\xi} d\eta \tag{95}$$

The integrals $g_2$ and $g_3$ are not singular when the source point is node 1 because $N_2 = N_3 = 0$ at node 1, where $T^* \to \infty$.

- Source point at node 2: $\xi_d = 0$.

$$r_A = \frac{\xi - \xi_d}{2} = \frac{\xi - 0}{2} = \frac{\xi}{2} \tag{96}$$

$$g_{2s} = \frac{-1}{2\pi k}\int_{-1}^{1}\log(\frac{\xi}{2})N_2\frac{d\Gamma}{d\xi}d\xi = \frac{-1}{2\pi k}\int_{-1}^{1}\log(\xi)N_2\frac{d\Gamma}{d\xi}d\,xi$$
$$-\frac{-1}{2\pi k}\int_{-1}^{1}\log(2)N_2\frac{d\Gamma}{d\,xi}d\xi = g_{2s1} + g_{2s2} \quad (97)$$

where

$$g_{2s2} = \frac{-1}{2\pi k}\int_{-1}^{1}\log(2)N_2\frac{d\Gamma}{d\xi}d\xi \quad (98)$$

is a regular integral and can be integrated using standard Gaussian quadrature.

$$g_{2s1} = \frac{-1}{2\pi k}\int_{-1}^{1}\log(\xi)N_2\frac{d\Gamma}{d\xi}d\xi \quad (99)$$

is an integral with weak singularity and must be calculated using logarithmic Gauss quadrature through the following transformation:

$$\eta = \xi \Rightarrow \frac{d\eta}{d\xi} = 1 \quad (100)$$

one has:

$$\eta(\xi = 0) = 0 \quad (101)$$

$$\eta(\xi = 1) = 1 \quad (102)$$

$$r_A = \frac{\xi - \xi_d}{2} = \frac{\xi}{2} = \frac{\eta}{2} \quad (103)$$

Hence, we have:

$$g_{2s1} = 2 \times \frac{-1}{2\pi k}\int_{0}^{1}\log\left(\frac{\eta}{2}\right)N_2(\xi(\eta))\frac{d\Gamma}{d\xi}\frac{d\xi}{d\eta}d\eta = \frac{-1}{\pi k}\int_{0}^{1}\log(\eta)\,N_2(\xi(\eta))\frac{d\Gamma}{d\xi}d\eta \quad (104)$$

The integrals $g_1$ and $g_3$ are not singular when the source point is node 2 because $N_1 = N_3 = 0$ at node 2, where $T^* \to \infty$.

- Source point at node 3: $\xi_d = 1$.

The integrals $g_1$ and $g_2$ are not singular when the source point is node 3 because $N_1 = N_2 = 0$ at node 3, where $T^* \to \infty$.

$$\int_{-1}^{1} N_1 T^* \frac{d\Gamma}{d\xi}d\xi\bigg|_{\xi_d=-1} = \int_{-1}^{1} N_3 T^* \frac{d\Gamma}{d\xi}d\xi\bigg|_{\xi_d=1} \quad (105)$$

In this way, the integral $g_3$ does not need to be calculated when the source point is node 3 because it uses the calculated value of the integral $g_1$ when the source point is node 1.

## 8 Heat sources

Given Laplace's equation for a heat conduction problem, as seen:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

with its respective integral boundary equation

$$cT(x_d, y_d) = \int_\Gamma q^* T d\Gamma - \int_\Gamma T^* q d\Gamma.$$

If there is heat generation, the Poisson formulation given by:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f(x, y), \tag{106}$$

where $f(x, y)$ is the heat generation function (heat source).

Multiplying Eq.(106) by a weight function and integrating along the boundary we obtain a residual function. Assuming that this residue is equal to zero, we have:

$$\int_A \left[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - f(x, y) \right] \omega dA = 0.$$

In order to obtain the integral boundary equation, it follows:

$$\int_A \left[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right] \omega dA - \int_A f(x, y) \omega dA = 0 \tag{107}$$

that works

$$cT(x_d, y_d) = \int_s q^* T ds - \int_s T^* q ds + \int_A T^* f(x, y) dA. \tag{108}$$

Observing that there is a domain integral in the formulation of Eq.(108) that has to be transformed into a boundary integral, otherwise, the problem domain will have to be discretized.

### 8.1 Concentrated heat sources

If the heat source is concentrated, it will be represented by a Dirac delta function, ie:

$$f(x, y) = C\delta(x - x_c, y - y_c) \tag{109}$$

where $(x_c, y_c)$ are the coordinates of the point where the applied heat source and $C$ is the value of the heat source. If $C$ is negative, $f(x, y)$ is a concentrated heat sink. Substituting the equation (109) into the equation (108) we have:

$$cT(x_d, y_d) = \int_s q^* T ds - \int_s T^* q ds + \int_A T^* C\delta(x - x_c, y - y_c) dA. \tag{110}$$

By the properties of the Dirac delta, the domain integral becomes the value of the function at the point, that is:

$$cT(x_d, y_d) = \int_s q^* T ds - \int_s T^* q ds + CT^*(x_c - x_d, y_c - y_d). \tag{111}$$

## 8.2 Domain distributed heat sources

If the heat source $f(x, y)$ is distributed in the domain, the radial integration method can be used to transform the domain integral into a boundary integral, as follows:

$$\int_A T^* f(x,y) dA = \int_{\theta_1}^{\theta_2} \underbrace{\int_0^r T^* f[x(rho,\theta), y(\rho,\theta)] \rho d\rho}_{F} \, d\theta.$$

Making

$$F = \int_0^r T^* f[x(\rho,\theta), y(\rho,\theta)] \rho d\rho, \tag{112}$$

results

$$\int_A T^* f(x,y) dA = \int_{\theta_1}^{\theta_2} F d\theta. \tag{113}$$

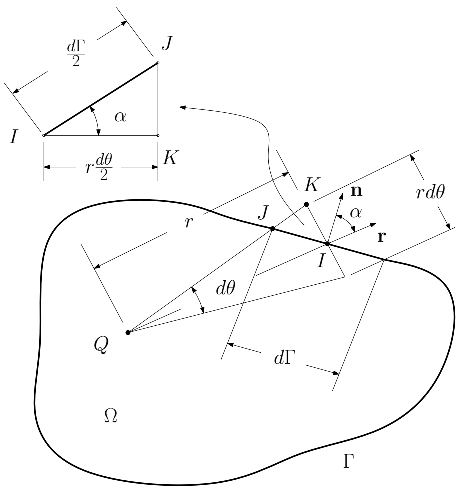


**Figure 10: Transformation from domain to boundary.**

From the triangle in the figure 10, it follows:

$$cos\alpha = \frac{r\frac{d\theta}{2}}{\frac{d\Gamma}{2}}$$

$$d\theta = \frac{cos\alpha}{r} ds. \tag{114}$$

Since $\vec{n}$ and $\vec{r}$ are unit vectors, we have:

$$cos\alpha = \vec{n}.\vec{r}. \tag{115}$$

Replacing Eq.(115) in Eq.(114) and then in Eq.(113), we have

$$\int_A T^* f(x,y) dA = \int_s F \frac{\vec{n}.\vec{r}}{r} ds. \tag{116}$$

Replacing Eq.(116) in Eq.(107), follows

$$cT(x_d, y_d) = \int_s q^* T ds - \int_s T^* q ds + \int_s F \frac{\vec{n}.\vec{r}}{r} ds,$$

which is the integral boundary equation when heat is generated.

## 9 Numerical examples

To evaluate the boundary element formulations using constant, linear and quadratic elements, the temperature distribution in a cylinder and a rectangular plate was analyzed. In the cylinder, the boundary conditions are constant in the outer and inner diameters, while in the plate, the boundary conditions vary at each point of the boundary.

**Example 9.1 Conduction of heat in a cylinder** *Consider a cylinder with dimensions shown in figure 11. The problem was discretized with different meshes, from the coarsest to the most refined. It was considered* $r_i = 1$, $r_o$*=2,* $T(r_i)$*= 100 and* $q(r_o)$*= -200,* $k$ *= 1.*

*The analytical solution for temperature is given by:*

$$T(r) = T(r_i) - q(r_o)\, r_o \log(r/r_i) \tag{117}$$

*and to the stream by:*

$$q(r) = -q(r_o)\frac{r_o}{r}. \tag{118}$$

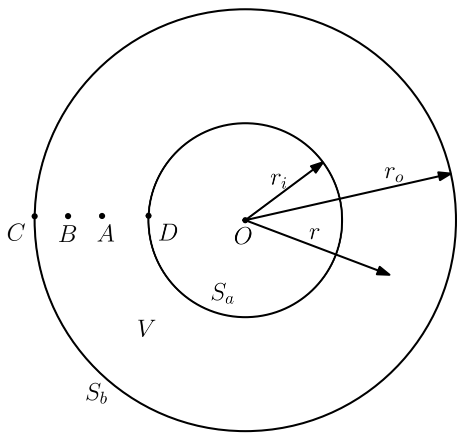


**Figure 11: Cylinder dimensions**

*The figures 12, 13 and 14 show, respectively, a 16-node mesh with constant boundary elements, a 112-node mesh with constant boundary elements, and a 16-node mesh with quadratic boundary elements. Note that, for a coarse discretization, with 16 nodes, the approximation of a circle with quadratic elements, which can be curved, is better satisfied than with straight elements (constant or linear elements).*

*Results were rated at 4 points. The first two points are the internal points* $A$ *and* $B$*, where* $r_A = (r_i + r_o)/2 = 1.50$ *and* $r_B = (r_i + 3r_o)/4 = 1.75$*. The two others are the boundary points* $C$ *and* $D$*. The value of temperature* $T$ *and flux* $q$ *at these points were calculated with different meshes and different types of elements and the results were compared with analytical solutions of the problem for temperature and flux, given by the equations (117 ) and (118), respectively. The figures 16, 17, 18, 19, 20 and 21 show these comparisons.*

*Care was taken that the number of nodes was the same in each comparison. For this, the number of quadratic elements was half the number of linear and constant elements. Furthermore, so that the accuracy of the integration did not influence the analysis, a*

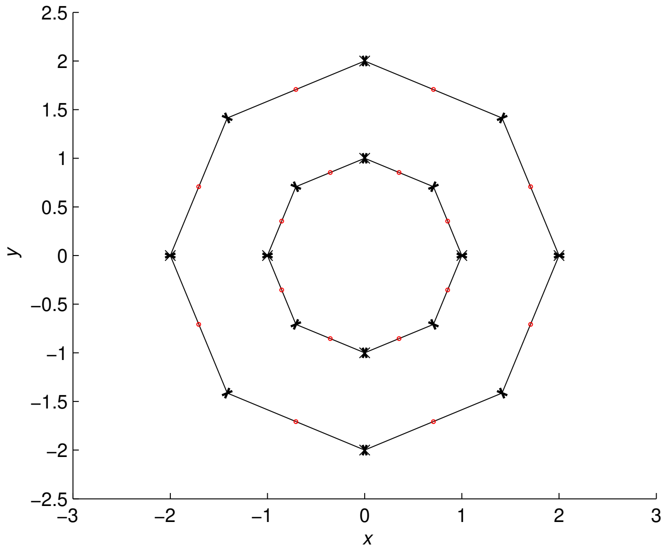


**Figure 12: 16-node boundary element mesh with constant elements (8 on the outer boundary and 8 on the inner boundary)**

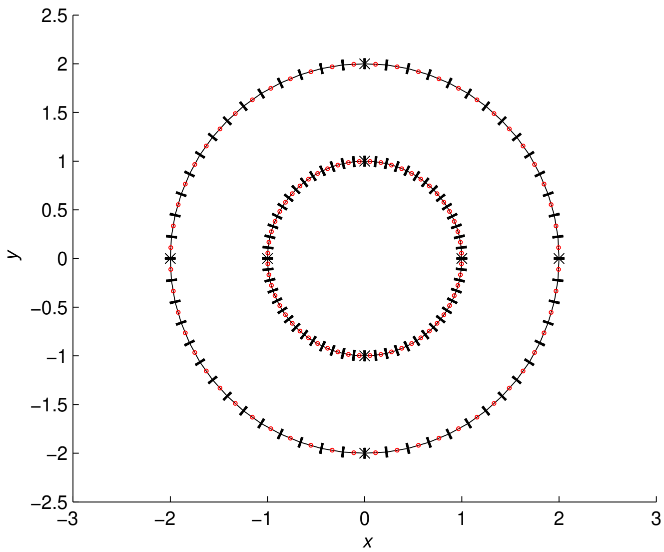


**Figure 13: Boundary element mesh with 112 nodes with constant elements (56 on the outer boundary and 56 on the inner boundary)**

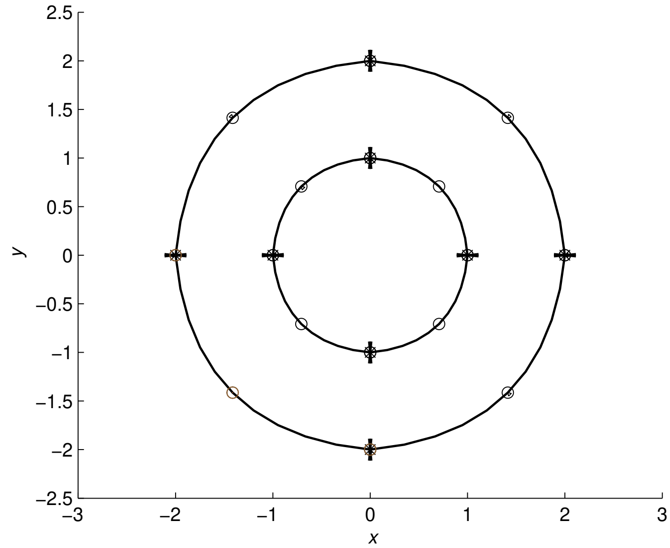


**Figure 14: Boundary element mesh with 16 nodes with quadratic elements (8 on the outer boundary and 8 on the inner boundary)**

*large number of integration points were used in all the integrals of the boundary elements method. All integrals were calculated with 16 Gauss points, which represents a number more than enough for an integration with good precision.*

*Analyzing the figures 16, 17, 18, 19, 20 and 21, it is noted that all formulations converge to the solution analytical as the mesh is refined. However, it was not possible to observe any element that showed faster convergence in all cases. For the temperature at the inner points $A$ and $B$, the quadratic elements showed the fastest convergence and the constant elements the slower convergence. For the flux at the inner points $A$ and $B$, the linear elements showed the slowest convergence while the constant elements converged faster at the point $A$ and the quadratic ones converge faster at the point $B$. At the points $C$ and $D$, belonging to the inner and outer boundaries, respectively, the results for quadratic elements were analyzed both at the element's endpoints and at the element's middle nodes. In the case of point $C$, where temperature was the unknown variable, the fastest convergence was presented by the constant elements while the slower one was presented by the linear elements. In the case of point $D$, where the flux was calculated, the fastest convergence was that of the constant elements, while the slowest was presented by the linear elements. In the last two cases the convergence to the grid of quadratic elements was faster at the middle nodes than at the endpoints of the elements.*

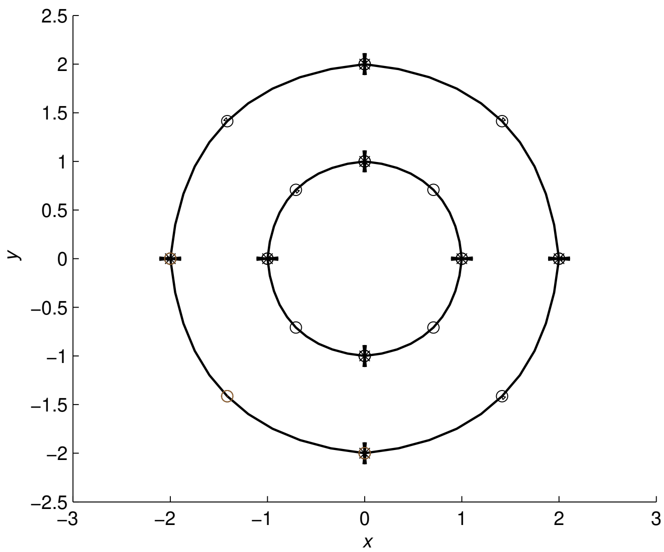


**Figure 15: 16-node boundary element mesh with quadratic elements (8 on the outer boundary and 8 on the inner boundary)**

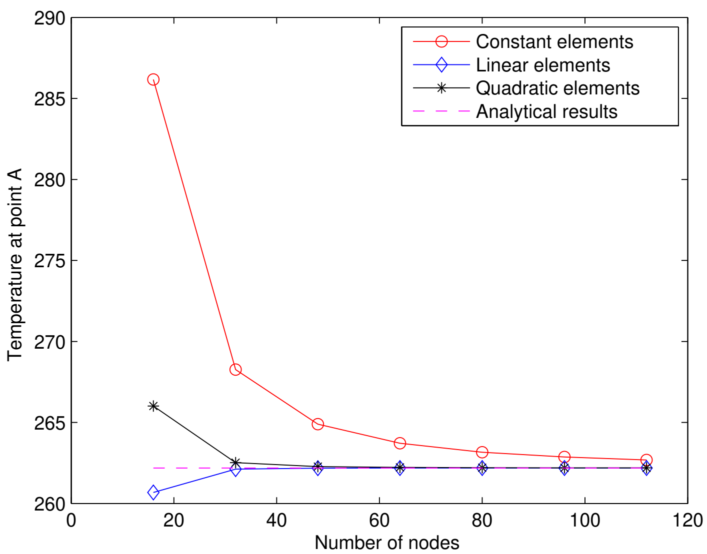


**Figure 16: Temperature at point $A$.**

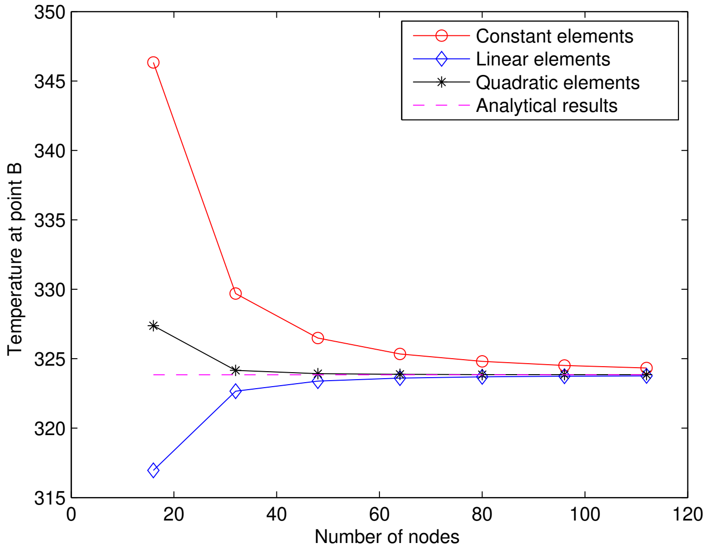


**Figure 17: Temperature at point $B$.**

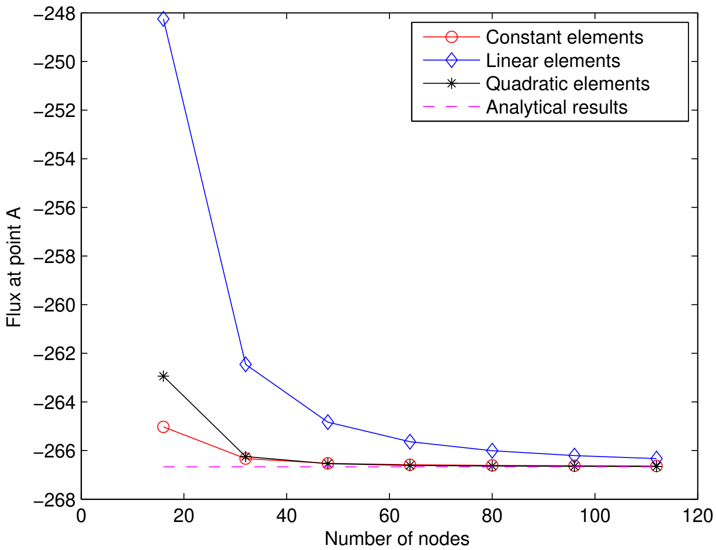


**Figure 18: flux at point $A$.**

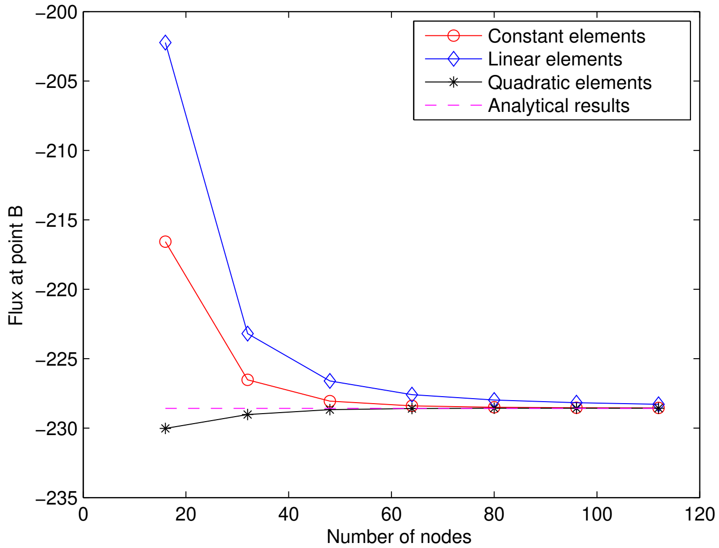


**Figure 19: flux at point $B$.**

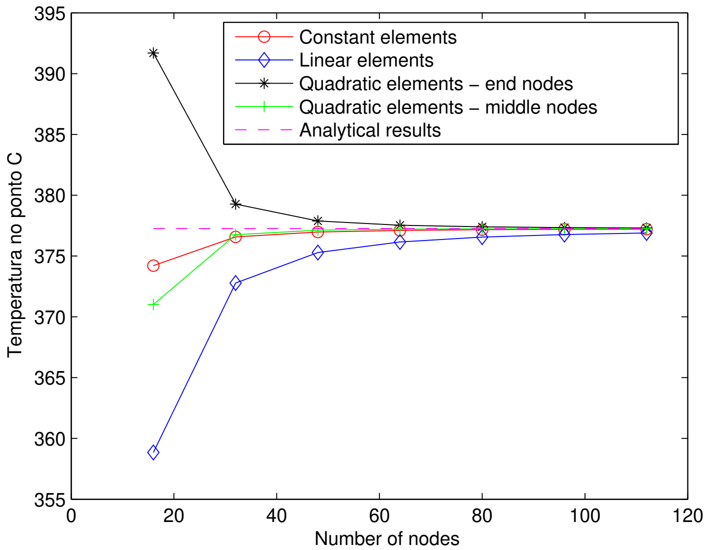


**Figure 20: Temperature at point $C$.**

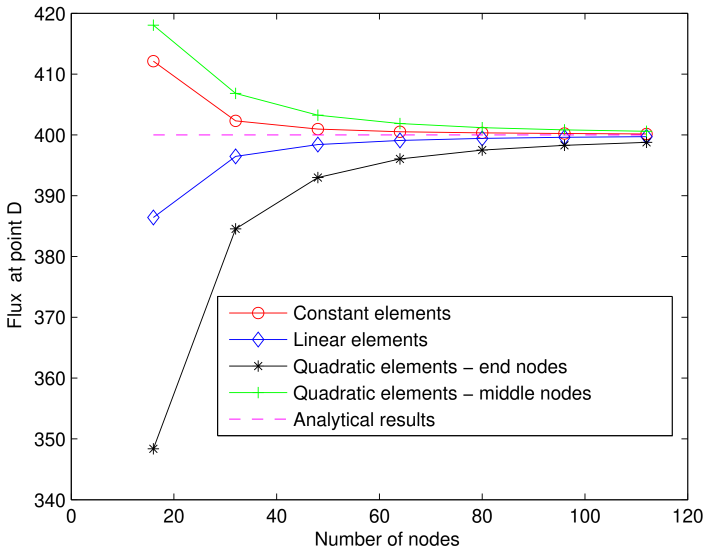


**Figure 21: flux at point $D$.**

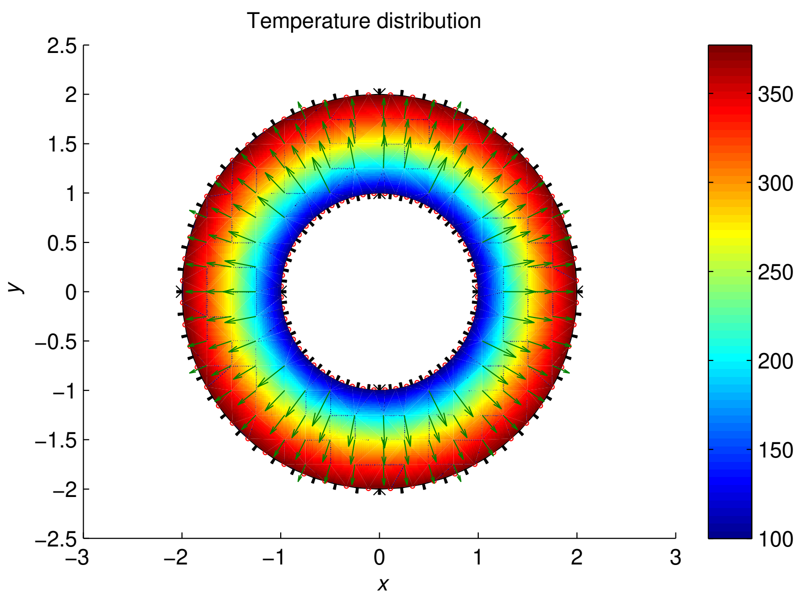


**Figure 22: Distribution of temperature and heat flux across the cylinder.**

**Example 9.2 Conduction of heat on a plate**

*Consider a rectangular plate $ABCD$ as shown in figure 23. It was considered $k$ = 1. The boundary conditions on the plate are as follows:*

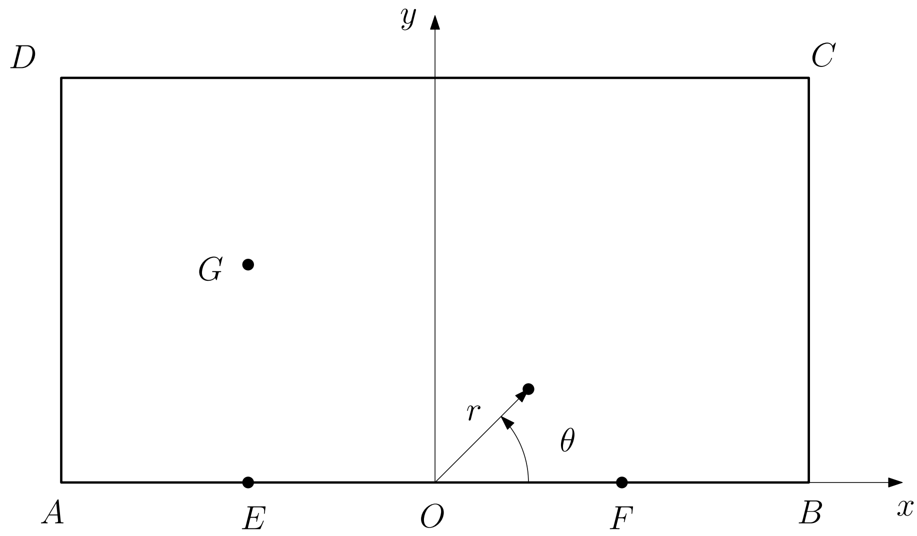


**Figure 23: Rectangular plate.**

$$q = -\frac{1}{2\sqrt{r}}\left(\cos\frac{\theta}{2}\cos\theta + \sin\frac{\theta}{2}\sin\theta\right) \quad \text{in} \quad BC, \tag{119}$$

$$q = -\frac{1}{2\sqrt{r}}\left(\cos\frac{\theta}{2}\sin\theta - \sin\frac{\theta}{2}\cos\theta\right) \quad \text{in} \quad CD, \tag{120}$$

$$q = \frac{1}{2\sqrt{r}}\left(\cos\frac{\theta}{2}\cos\theta + \sin\frac{\theta}{2}\sin\theta\right) \quad \text{in} \quad DA, \tag{121}$$

$$T = 0 \quad \text{in} \quad AO \tag{122}$$

*and*

$$q = 0 \quad \text{in} \quad OB. \tag{123}$$

*The analytical solution to this problem is given by:*

$$u = \sqrt{r}\cos\frac{\theta}{2}, \tag{124}$$

$$q_x = \frac{\cos\frac{\theta}{2}}{2\sqrt{r}} \tag{125}$$

*and*

$$q_y = \frac{\sin\frac{\theta}{2}}{2\sqrt{r}}. \tag{126}$$

*The coordinates of the points $A$ and $C$ are, respectively (-1.0; 0.0) and (1.0; 1.0). The $E$ point is the midpoint of the $AO$ segment and the $F$ point is the midpoint of the $OB$ segment. The point $G$ has coordinate (-0.5; 0.5).*

*As in the previous example, the rectangular plate was also discretized with different meshes, from the coarsest (24 knots) to the most refined (120 knots). In all cases, the elements used were close in size but not equal in size. The temperature value $T$ was calculated at points $F$ and $G$, the flux normal to the boundary $q$ was calculated at point $E$ and fluxs $q_x$ and $q_y$, in the directions $x$ and $y$, respectively, were calculated at point $G$. The figures 24, 25, 26, 27, 28 show the values of temperatures and fluxs at these points.*

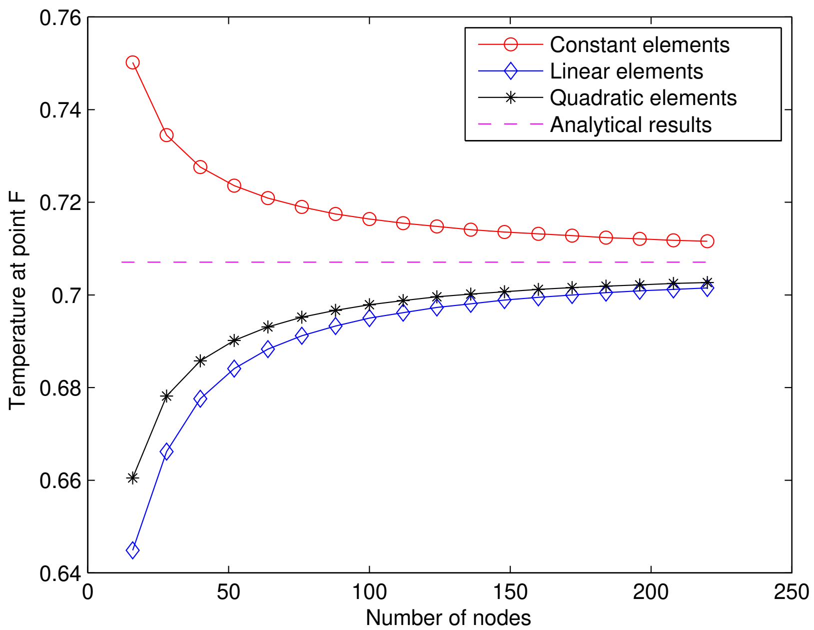


**Figure 24: Temperature at point $F$.**

*Figure 29 shows the temperature distribution and heat flux in the rectangular plate.*

*The behavior of the results obtained in this example are, in most cases, very similar to the behavior obtained in the previous example. All formulations converge into the analytical solution at all points for both temperature and flux. The linear elements showed a slightly slower convergence than the quadratic and constant elements, the latter two having very similar convergence, although they approach the analytical solution from opposite sides (one above and one below the analytical solution ). In the case of Figure 25, the quadratic and linear elements presented expressive oscillations for the coarser meshes, which stabilized with the refinement of the mesh. These oscillations also occurred in a less expressive way in Figure 28.*

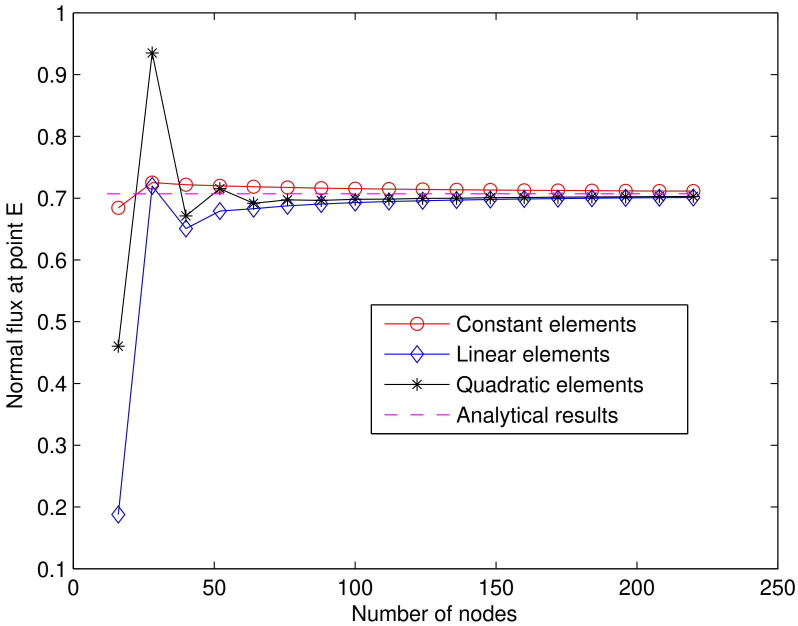


**Figure 25: Heat flux normal to the boundary at point $E$.**

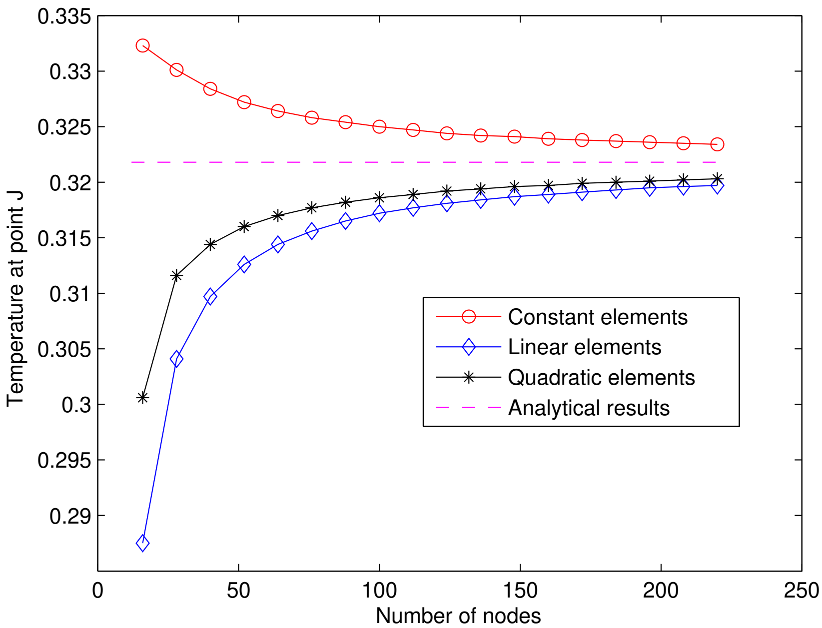


**Figure 26: Temperature at point $G$.**

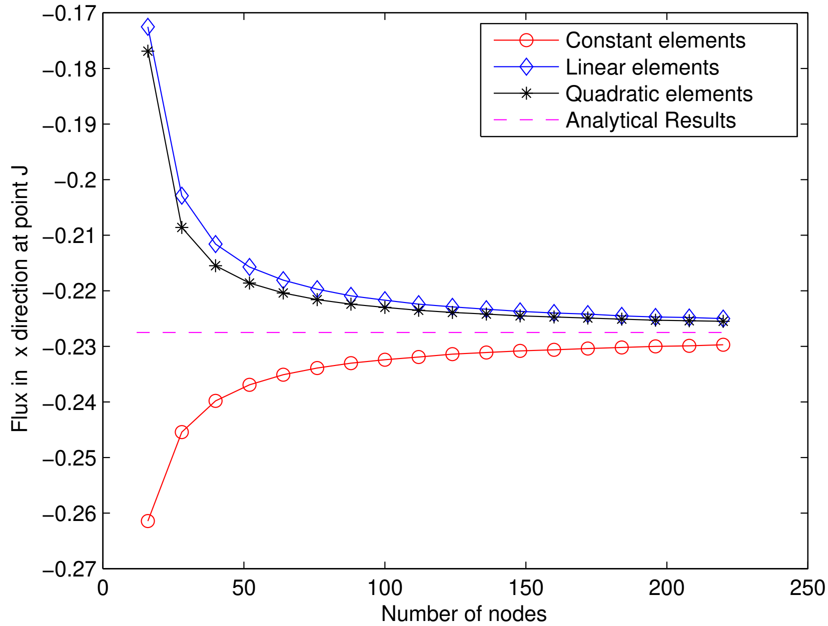


**Figure 27: Heat flux in the direction $x$ at point $G$.**

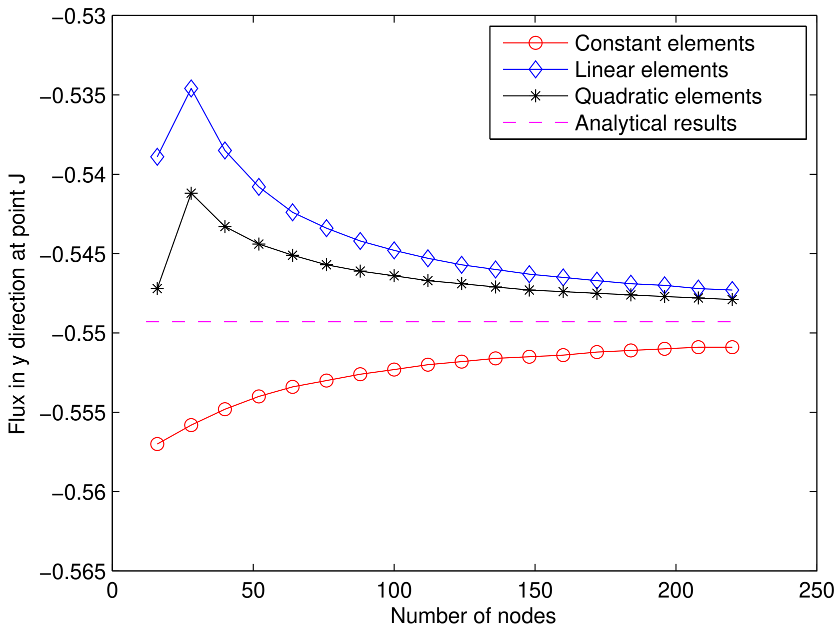


**Figure 28: Heat flux in the $y$ direction at the point $G$.**

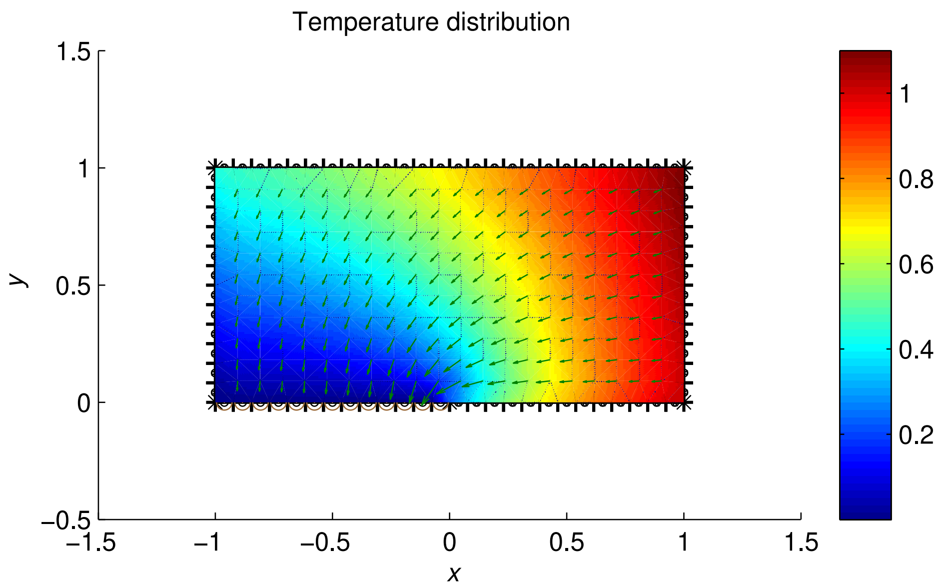


**Figure 29: Distribution of temperature and heat flux on the plate.**

## 10 Concluding remarks

For certain problems, if an appropriate auxiliary field, known as the fundamental solution for the problem, is used in the integral formulation, the domain integrals may be resolved and rewritten as non-integral terms, thus leading to an integral representation wherein only boundary integrals remain. The equation obtained from the use of integral identities and fundamental solutions is called the boundary integral equation (BIE). The fundamental solution is an auxiliary field that satisfies the partial differential equation for the problem being modeled, and is valid for an extended, infinite domain that surrounds and encompasses the domain of the BVP being studied. For a particular problem, the fundamental solution is a function of the distance between two different points in the extended domain, among other possible variables. The usual way to understand the fundamental solution is to let an observer sit in one of these points, called the collocation point, while the other point, called the field point, is allowed to vary throughout the extended domain. When the distance between these two points becomes zero, the fundamental solution is singular. As the fundamental solution is part of the integrand of the BIE, the solution of this boundary integral representation requires the understanding of singular integrals, which arise when the collocation and field points coincide. One must note that although a mathematical singularity will appear due to the use of fundamental solutions, the original variable of interest represents a field that usually is not physically singular, and thus, although the integrand may be singular, the integral exists and has a non singular value, in this case. A numerical solution for a particular BIE, containing boundary integrals only, can be performed through the discretization of the domain boundary into boundary elements. To perform this discretization, interpolation functions need to be used for each boundary element, both for the geometry and for the unknown variables of the problem. A usual approach is to use polynomial interpolation functions similar to the ones adopted for the finite element method (FEM). Again, the interpolation functions can be expressed in terms of shape functions (isoparametric elements). After the discretization of the boundary, the integrals can be numerically solved by some quadrature scheme, to obtain an algebraic equation in terms of the boundary variables at the nodes. The numerical integration for non-singular integrals may follow standard Gauss quadrature procedures, for example, but the integration of the singular integrals may require some special techniques.

When collocation is performed for one particular point at the boundary, only one integral equation is obtained. The discretization of this integral equation leads to only one algebraic equation written in terms of the quantities of interest at the boundary nodes. Collocating at a different boundary point leads to another boundary integral equation, which is independent from the equation obtained from the collocation at the first boundary point. But, for the discretized problem, for a given number of boundary unknowns at the nodes (for example, n nodes with one unknown per node, leading to a total of n boundary unknowns) only the first set of discretized algebraic equations (in this example, the first n algebraic equations), obtained by changing the collocation point from one boundary location to another, will consist of independent equations. Thus, to 99 be able to obtain a system of linearly independent equations to be solved, one must collocate at a number of different boundary points, so that the matrix of the system of equations is a square, invertible matrix. Although collocation could have been performed at any boundary point, an usual procedure is to collocate at the boundary nodes, so that all the boundary information necessary to solve the problem is represented in terms of the boundary nodal variables. Usually, there are more boundary unknowns than available independent algebraic equations obtained from this approach. Only by imposing the boundary conditions, for the prescribed boundary quantities, the system of equations could be determined, for the remaining boundary unknowns.

After obtaining the unknown boundary quantities of interest as the solution of the system of equations, a post-processing approach can be used, performing further collocation at any other points, either in the domain (interior points) our outside the domain (exterior points), to obtain the quantities of interest at these points. The above-outlined method was originally known as the Boundary Integral Equation (BIE) method, and later has become to known as the Boundary Element Method (BEM). Some important points must be made:

- When the boundary integral equation is written for collocation points outside the boundary (interior or exterior points), as the field point remains a boundary point, no singular integrals are obtained for collocation at these interior or exterior points.

- One must note that, differently from the FEM, although element interpolating functions are being used for each boundary element, this numerical approach does not lead to local support, as the auxiliary function being used is a fundamental solution, which is eminently non-local. The fundamental solution requires information from a field point, which belongs to the element being integrated, and from a collocation point, which may be in the element being integrated, or may be at some other location elsewhere in the boundary, thus outside this element. Thus, the matrix of the system of equations, obtained after the discretization of the boundary and after imposing the boundary conditions, is non-symmetric and fully-populated.

- Every boundary integral equation obtained from collocation at a particular point is an exact equation, as both the proper integral identity and the fundamental solution used are, in fact, exact representations for the problem being modeled. In the BEM, the only approximation occurs when the boundary is discretized into boundary elements. Therefore, in the BEM, exact integral equations are written for the approximate boundary, while in the FEM, an approximate integral representation in the domain was used, and another approximation was also used in every finite element, as the geometry and the quantities of interest were described in terms of the interpolating functions.

For further reading on the Boundary Element Method for potential, fluids, acoustics, elasticity, plates, and shells, some books ate indicated on references Paris et al. [1997], Beer et al. [2008], Gaul et al. [2003], Wrobel [2002], Aliabadi [2002], Katsikadelis [2002], Bonnet [1999], Beer and Watson [1992], Jaswon [1977].

## References


Aliabadi. *Boundary Element Method V 2*. John Wiley & Sons, Apr. 2002. ISBN 0470842989.

G. Beer and J. Watson. *Introduction to Finite and Boundary Element Methods for Engineers*. Wiley, 1992. ISBN 0 471 92813 5.

G. Beer, I. Smith, and C. Duenser. *The Boundary Element Method with Programming*. Springer Vienna, Apr. 2008. ISBN 3211715746.

M. B. Bonnet. *Boundary Integral Equation Methods*. John Wiley & Sons, June 1999. ISBN 0471971847.

C. Brebbia and J. Dominguez. *Boundary Element an Introductory Course*. Computation Mechanics Publications, Southampton, Boston, second edition, 1992.

L. Gaul, M. KÃ¶gl, and M. Wagner. *Boundary Element Methods for Engineers and Scientists*. Springer Berlin Heidelberg, Feb. 2003. ISBN 3540004637.

M. A. Jaswon. *Integral equation methods in potential theory and elastostatics (Computational mathematics and applications)*. Academic Press, 1977. ISBN 9780123810502.

J. T. Katsikadelis. *Boundary Elements: Theory and Applications*. Elsevier Science & Techn., May 2002. ISBN 9780080528243.

F. Paris, J. Canas, and J. Cañas. *Boundary element method: fundamentals and applications*. Oxford University Press, USA, 1997.

J. C. F. Telles. A self adptive co-ordinate transformation for efficient numerical evaluation of general boundary element integrals. *International Journal for Numerical Methods in Engineering*, 24:959–973, 1987.

L. C. Wrobel. *Boundary Element Method V 1*. John Wiley & Sons, Apr. 2002. ISBN 0471720399.